CS 573: Algorithms, Fall 2009

Homework 0, due September 2, 23:59:59, 2009

Name:	
Net ID:	Alias:

Neatly print your name (first name first, with no comma), your network ID, and a short alias into the boxes above. **Do not sign your name. Do not write your Social Security number.** Staple this sheet of paper to the top of your homework.

Grades will be listed on the course web site by alias give us, so your alias should not resemble your name or your Net ID. If you don't give yourself an alias, we'll give you one that you won't like.

This homework tests your familiarity with the prerequisite material from CS 173, CS 225, and CS 273—many of these problems have appeared on homeworks or exams in those classes—primarily to help you identify gaps in your knowledge. **You are responsible for filling those gaps on your own.** Chapters 1–6 of CLR should be sufficient review, but you may want to consult other texts as well.

Before you do anything else, read the Homework Instructions and FAQ on the CS 573 course web page (`http://www.cs.uiuc.edu/class/fa09/cs573/faq.html`), and then check the box below. This web page gives instructions on how to write and submit homeworks—staple your solutions together in order, write your name and netID on every page, don't turn in source code, analyze everything, use good English and good logic, and so forth.

☐ I have read the CS 573 Homework Instructions and FAQ.

> "Be that as it may, it is to night school that I owe what education I possess; I am the first to own that it doesn't amount to much, though there is something rather grandiose about the gaps in it." – The tin drum, Gunter Grass

Required Problems

1. **[10 Points]** Prove that for any nonnegative integer parameters a and b, the following algorithms terminate and produce identical output. Also, provide bounds on the running times of those algorithms. Can you imagine any reason why WeirdEuclid would be preferable to FastEuclid?

```
SlowEuclid(a, b) :
    if b > a
        return SlowEuclid(b, a)
    else if b = 0
        return a
    else
        return SlowEuclid(b, a − b)
```

```
FastEuclid(a, b) :
    if b = 0
        return a
    else
        return FastEuclid(b, a mod b)
```

```
WeirdEuclid(a, b) :
    if b = 0
        return a
    if a = 0
        return b
    if a is even and b is even
        return 2∗WeirdEuclid(a/2, b/2)
    if a is even and b is odd
        return WeirdEuclid(a/2, b)
    if a is odd and b is even
        return WeirdEuclid(a, b/2)
    if b > a
        return WeirdEuclid(b − a, a)
    else
        return WeirdEuclid(a − b, b)
```

2. **Recurrences**
   [20 Points]

   Solve the following recurrences. State tight asymptotic bounds for each function in the form $\Theta(f(n))$ for some recognizable function $f(n)$. You do not need to turn in proofs (in fact, please *don't* turn in proofs), but you should do them anyway just for practice. Assume reasonable but nontrivial base cases if none are supplied. More exact solutions are better.

   (a) **[2 Points]** $A(n) = A\Big((n-5)^{1/4} + \lfloor \log^2 n \rfloor\Big) + n.$

   (b) **[2 Points]** $B(n) = \min_{0<k<n/2} \big(1 + B(n/2+k) + B(n/2-k)\big).$

   (c) **[2 Points]** $C(n) = 12C(\lceil n/9 \rceil + 5) + n/\log n.$

   (d) **[2 Points]** $D(n) = \frac{n-4}{n} D(n-2) + 1.$

   (e) **[2 Points]** $E(n) = E(\lfloor 3n/7 \rfloor) + \sqrt{n \log n}.$

   (f) **[2 Points]** $F(n) = F(\sqrt{\log n}) + \log n.$

   (g) **[2 Points]** $G(n) = n + \lfloor n^{1/4} \rfloor \cdot G(\lfloor n^{3/4} \rfloor)$

   (h) **[2 Points]** $H(n) = \log(H(n-9)) + \log^* n.$

   (i) **[2 Points]** $I(n) = 7I\big(\lfloor n^{1/6} \rfloor\big) + 1.$

(j) **[2 Points]** $J(n) = 4J(n/7) + 1$

3. SORTING FUNCTIONS
[20 Points]

Sort the following 25 functions from asymptotically smallest to asymptotically largest, indicating ties if there are any. You do not need to turn in proofs (in fact, please *don't* turn in proofs), but you should do them anyway just for practice.

$n^{5.5} - (n-1)^{5.5}$	n	$n^{2.2}$	$\lg^*(n/7)$	$1 + \lg\lg n$
$\cos n + 2$	$\lg(\lg^* n)$	$\lg(n!)$	$(\lg^* n)^{\lg n}$	n^4
$\lg^* 2^{2^n}$	$2^{\lg^* n}$	$e^{\sqrt{n}}$	$\sum_{i=1}^{n} \frac{1}{i}$	$\sum_{i=1}^{n} \frac{1}{i^2}$
$n^{\frac{3}{(2n)}}$	$n^{3/(2\lg n)}$	$\lfloor \lg\lg(n!) \rfloor$	$(\lg(7+n))^{\lg n}$	$\left(1 + \frac{1}{154}\right)^{154n}$
$n^{1/\lg\lg n}$	$n^{\lg\lg n}$	$\lg^{(200)} n$	$n^{1/1234}$	$n(\lg n)^3$

To simplify notation, write $f(n) \ll g(n)$ to mean $f(n) = o(g(n))$ and $f(n) \equiv g(n)$ to mean $f(n) = \Theta(g(n))$. For example, the functions n^2, n, $\binom{n}{2}$, n^3 could be sorted either as $n \ll n^2 \equiv \binom{n}{2} \ll n^3$ or as $n \ll \binom{n}{2} \equiv n^2 \ll n^3$.

4. **[20 Points]** There are n balls (numbered from 1 to n) and n boxes (numbered from 1 to n). We put each ball in a randomly selected box.

   (a) **[4 Points]** A box may contain more than one ball. Suppose X is the number on the box that has the smallest number among all nonempty boxes. What is the expectation of X?

   (b) **[4 Points]** What is the expected number of bins that have exactly one ball in them? (Hint: Compute the probability of a specific bin to contain exactly one ball and then use some properties of expectation.)

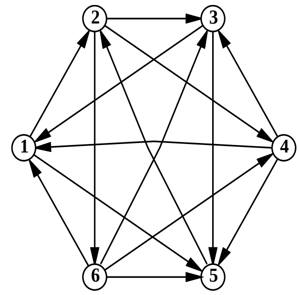
   (c) **[8 Points]** We put the balls into the boxes in such a way that there is exactly one ball in each box. If the number written on a ball is the same as the number written on the box containing the ball, we say there is a match. What is the expected number of matches?

   (d) **[4 Points]** What is the probability that there are exactly k matches? ($1 \le k < n$)

   [Hint: If you have to appeal to "intuition" or "common sense", your answers are probably wrong!]

5. A TRIP THROUGH THE GRAPH.
[20 Points]

A *tournament* is a directed graph with exactly one edge between every pair of vertices. (Think of the nodes as players in a round-robin tournament, where each edge points from the winner to the loser.) A *Hamiltonian path* is a sequence of directed edges, joined end to end, that visits every vertex exactly once. Prove that every tournament contains at least one Hamiltonian path.



A six-vertex tournament containing the Hamiltonian path $6 \to 4 \to 5 \to 2 \to 3 \to 1$.