# Parallel Numerical Algorithms Chapter 11 – QR Factorization

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## **QR** Factorization

For given m × n matrix A, with m > n, QR factorization has form

$$oldsymbol{A} = oldsymbol{Q} egin{bmatrix} R \ O \end{bmatrix}$$

where matrix  ${\bm Q}$  is  $m\times m$  and orthogonal, and  ${\bm R}$  is  $n\times n$  and upper triangular

- Can be used to solve linear systems, least squares problems, etc.
- As with Gaussian elimination, zeros are introduced successively into matrix *A*, eventually reaching upper triangular form, but using orthogonal transformations instead of elementary eliminators

Methods for QR Factorization

- Householder transformations (elementary reflectors)
- Givens transformations (plane rotations)
- Gram-Schmidt orthogonalization

Householder Transformations

• Householder transformation has form

$$\boldsymbol{H} = \boldsymbol{I} - 2 \frac{\boldsymbol{v} \boldsymbol{v}^T}{\boldsymbol{v}^T \boldsymbol{v}}$$

where v is nonzero vector

- From definition, H = H<sup>T</sup> = H<sup>-1</sup>, so H is both orthogonal and symmetric
- For given vector a, choose v so that

$$\boldsymbol{H}\boldsymbol{a} = \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \alpha \boldsymbol{e}_1$$

### Householder Transformations

• Substituting into formula for H, we see that we can take

$$\boldsymbol{v} = \boldsymbol{a} - \alpha \boldsymbol{e}_1$$

and to preserve norm we must have  $\alpha = \pm \|a\|_2$ , with sign chosen to avoid cancellation

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## Householder QR Factorization

for 
$$k = 1$$
 to  $n$   
 $\alpha_k = -\operatorname{sign}(a_{kk})\sqrt{a_{kk}^2 + \dots + a_{mk}^2}$   
 $v_k = \begin{bmatrix} 0 & \cdots & 0 & a_{kk} & \cdots & a_{mk} \end{bmatrix}^T - \alpha_k e_k$   
 $\beta_k = v_k^T v_k$   
if  $\beta_k = 0$  then  
continue with next  $k$   
for  $j = k$  to  $n$   
 $\gamma_j = v_k^T a_j$   
 $a_j = a_j - (2\gamma_j/\beta_k)v_k$   
end  
end

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### Parallel Householder QR

- Householder QR factorization is similar to Gaussian elimination for LU factorization
- Forming Householder vector  $v_k$  is analogous to computing multipliers in Gaussian elimination
- Subsequent updating of remaining unreduced portion of matrix is also analogous to Gaussian elimination
- Thus, parallel implementation is similar to parallel LU, but with Householder vectors broadcast horizontally instead of multipliers
- For this reason, we will not go into details

#### **Givens Rotations**

- Givens rotation operates on pair of rows to introduce single zero
- For given 2-vector  $\boldsymbol{a} = [a_1 \ a_2]^T$ , if

$$c = \frac{a_1}{\sqrt{a_1^2 + a_2^2}}, \qquad s = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}$$

then

$$\boldsymbol{G}\boldsymbol{a} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

• Scalars c and s are cosine and sine of angle of rotation, and  $c^2 + s^2 = 1$ , so G is orthogonal

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### **Givens QR Factorization**

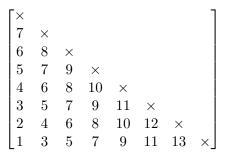
- Givens rotations can be systematically applied to successive pairs of rows of matrix *A* to zero entire strict lower triangle
- Subdiagonal entries of matrix can be annihilated in various possible orderings (but once introduced, zeros should be preserved)
- Each rotation must be applied to all entries in relevant pair of rows, not just entries determining *c* and *s*
- Once upper triangular form is reached, product of rotations, *Q*, is orthogonal, so we have QR factorization of *A*

## Parallel Givens QR Factorization

- With 1-D partitioning of *A* by columns, parallel implementation of Givens QR factorization is similar to parallel Householder QR factorization, with cosines and sines broadcast horizontally and each task updating its portion of relevant rows
- With 1-D partitioning of *A* by rows, broadcast of cosines and sines is unnecessary, but there is no parallelism unless multiple pairs of rows are processed simultaneously
- Fortunately, it is possible to process multiple pairs of rows simultaneously without interfering with each other

Parallel Givens QR Factorization

• Stage at which each subdiagonal entry can be annihilated is shown here for  $8 \times 8$  example



• Maximum parallelism is n/2 at stage n-1 for  $n \times n$  matrix



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# Parallel Givens QR Factorization

- Communication cost is high, but can be reduced by having each task initially reduce its entire local set of rows to upper triangular form, which requires no communication
- Then, in subsequent phase, task pairs cooperate in annihilating additional entries using one row from each of two tasks, exchanging data as necessary
- Various strategies can be used for combining results of first phase, depending on underlying network topology
- With hypercube, for example, final upper triangular form can be reached in log *p* combining steps



### Parallel Givens QR Factorization

- With 2-D partitioning of A, parallel implementation combines features of 1-D column and 1-D row algorithms
- In particular, sets of rows can be processed simultaneously to annihilate multiple entries, but updating of rows requires horizontal broadcast of cosines and sines

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