Please read the comments part of file lambda_maneu, which is required reading for this lecture and is available in the course web page. In particular, I assume familiarity with: (i) the signature $\Sigma_L$ defined there, and (ii) the equations

$$E_0 = \{ (U = \lambda z. (U z)) \mid U \in \overline{T}_{\Sigma_L} \land (\text{LAMBDABETAETA}-\text{ETA}(V)) \}$$

$$\vdash z = \text{fresh}(fU(V)) \land z \in \overline{T}_{\Sigma_L}$$

which I will abbreviate in what follows to $V_0$. I will write $z = \text{fresh}(fU(V))!S$, where $S$ are the equations for $fU$ and $[\Sigma_L]$. Recall also,

$$E_1 = \{ (U = W) \mid U,W \in \overline{T}_{\Sigma_L} \land (U \uparrow \beta \downarrow W) \}$$

which I will abbreviate in what follows to $F_1$. Recall, finally, that the $\Sigma_L$-algebra $\Lambda$ is defined as

$$\Lambda = \overline{T}_{\Sigma_L}/E_1 = \overline{T}_{\Sigma_L}/F_1. \text{ Likewise, for } G \text{ any set of }$$

$\Sigma_L$-equations we define $\Lambda G = \overline{T}_{\Sigma_L}/F_1\circ G$.

Recall, finally, that $\Lambda G$ is called extensional iff the function $[\text{E-1}] : \Lambda G \rightarrow [\Lambda G \rightarrow \Lambda G]$ is injective, where $[\text{E-1}]$ is the carrying of the operation $(-)G : \Lambda G \times \Lambda G \rightarrow \Lambda G$.

The point of this Addendum is to show that the proof of the Extensionality Theorem given in 3/4/16 actually proves a stronger result:

**Theorem (Extensionality).** Let $G$ be a set of $\Sigma_L$-equations. Then,

$$\Lambda G \text{ is extensional iff } \Lambda G \vdash \eta.$$
Proof

($\Rightarrow$) Suppose $\Lambda G$ is extensional. We have to prove that for each $[U]_G$ we have $[U]_G = [\lambda z.(U z)]_G$, where $z = \text{fresh}(\text{fr}(U))$. But since for any $[V]_G \in \Lambda G$ we have $(\lambda z.(U z)) V \rightarrow U V$, we then have $[U V]_G = [((\lambda z.(U z)) V)]_G$ which, by $\Lambda G$ being extensional forces $[U]_G = [\lambda z.(U z)]_G$, as desired.

($\Leftarrow$) Suppose $\Lambda G \vdash \mathcal{N}$. We have to prove for any $[M]_G, [M']_G \in \Lambda G$

that $\oplus \Rightarrow \ominus$

$\ominus$ $(\forall [N]_G \in \Lambda G). [M N]_G = [M' N]_G$

$\oplus$ $[M]_G = [M']_G$.

Choose $z \notin \text{fr}(M M')$. By $\ominus$ we have $[M z]_G = [M' z]_G$. But then we have

$$[M]_G = [\lambda z.(M z)]_G = [\lambda z.(M' z)]_G = [M']_G$$

by $\alpha + \eta$ by congruence by $\eta + \eta$

as desired. $\Box$

(Extensional and)

Corollary. $\Lambda G$ is initial in the class of all extensional $\Sigma_\Lambda$-algebras

of the form $\wedge G$:

$$
\begin{array}{c}
\Lambda \\
\rightarrow
\end{array}
\begin{array}{c}
\Lambda \cap \\
\cap
\end{array}
\begin{array}{c}
\Lambda G
\end{array}$$