Problem 1. Call a functor \( U : \mathcal{C} \to \mathcal{D} \) a right adjoint if there is a function \( F : \mathcal{D} \to \mathcal{C} \), and a family of arrows \( \eta = \{ A \to \eta_A : A \in \mathcal{D} \} \) such that \((U, F, \eta)\) is a right adjoint.

As it has been emphasized repeatedly in the lectures, the "internals" of how a given construction, i.e., a functor, are represented are immaterial, i.e., such constructions should be understood up to a natural change of representation. In particular, \( F \) above should be uniquely determined up to a natural change of representation.

(a) Prove that if \( U : \mathcal{C} \to \mathcal{D} \) is a right adjoint, then if \((U, F, \eta)\) and \((U, F', \eta')\) are right adjoints, then there is a natural equivalence \( \alpha : F \cong F' \) such that

\[
\begin{array}{ccc}

F_U & \cong & F'_U \\
\alpha_U \downarrow & & \downarrow \alpha' \\
\eta & \cong & \eta'
\end{array}
\]

(b) Give at least two pairs of examples of left adjoints \( F, F' \), both for the same \( U : \mathcal{C} \to \mathcal{D} \).
(c) Dualize the definition of \( U : \mathcal{C} \to \mathcal{D} \) right adjoint to a definition of \( F : \mathcal{D} \to \mathcal{C} \) left adjoint, and then dualize (a) to state precisely and prove that right adjoints to a left adjoint \( F : \mathcal{D} \to \mathcal{C} \) are unique up to natural equivalence.

**Problem 2**

An enormous economy of thought would be achieved if we were to know that if \( U : \mathcal{C} \to \mathcal{D} \) is a right adjoint and \( \mathcal{C} \xrightarrow{G} \mathcal{C} \) and \( \mathcal{D} \xrightarrow{H} \mathcal{D} \) are category isomorphisms, then the functor \( GUH^{-1} \)

\[
\begin{array}{c}
\mathcal{C} \\
\downarrow G
\end{array}
\xrightarrow{GUH^{-1}}
\begin{array}{c}
\mathcal{C} \\
\downarrow H^{-1}
\end{array}
\]

is also a right adjoint, because then a single proof that \( U \) is a right adjoint would automatically show that any such \( GUH^{-1} \) is also a right adjoint.

(a) Prove that if \( U : \mathcal{C} \to \mathcal{D} \) and \( F : \mathcal{D} \to \mathcal{C} \)

are functors, \( \eta : 1_{\mathcal{D}} \to FU \), \( \varepsilon : UF \to 1_{\mathcal{C}} \)
are natural transformations, and $G : C \to C'$, $H : D \to D'$ are isomorphisms of categories, then

$$(U, F, η, ε) \text{ is an adjunction i f } (GUH, H^{-1}FG^{-1}, H^{-1}ηH, GεG^{-1}) \text{ is an adjunction.}$$

(b) Note that we get a very useful corollary of (a):

If $U : C \to D$ is a right adjoint, and

$G : C \to C'$ is an isomorphism of categories, then

$GU : C' \to D$ is also a right adjoint.

Apply this to get the benefits of "killing four kinds with one stone" for the five categories:

1. $[M \to \text{Set}]$ (M a monoid)
2. $\text{RAct}_M$
3. $\text{Alg}(Σ_M, ζ_M)$
4. $\text{DAut}_X$

where (1)–(3) were proven isomorphic in Problem 2 of Homework 1, and for $M = X^k$ also isomorphic to (4).

Specifically,

(b.1) Prove that $U : \text{RAct}_M \to \text{Set}$
in a right adjoint to derive correspondingly right adjoints \( U': [M\to \text{Set}] \to \text{Set}, \ U'': \mathbb{A}_{(\mathbb{M}, \mathbb{E})} \to \text{Set} \) and \( U''': \text{DAut}_x \to \text{Set} \). Describe what these functors are, and what are their correspondingly left adjoints, guaranteed to exist by (a) above.

**Problem 3 (Products)**

Definition. A category \( \mathcal{C} \) is said to have (binary) products iff for any two objects \( A, B \in \mathcal{C} \) there is a third object, denoted \( A \times B \in \mathcal{C} \) and two arrows \( p_1: A \times B \to A \), \( p_2: A \times B \to B \), such that, given any object \( C \in \mathcal{C} \), and any pair of arrows \( f: C \to A \), \( g: C \to B \) there is a unique arrow \( \langle f, g \rangle : C \to A \times B \) such that

\[
\begin{array}{ccc}
C & \xrightarrow{\langle f, g \rangle} & A \times B \\
\downarrow f & & \downarrow p_1 \\
A & \xleftarrow{p_1} & A \times B \\
\end{array}
\]

\[
\begin{array}{ccc}
C & \xrightarrow{g} & B \\
\downarrow g & & \downarrow p_2 \\
A \times B & \xleftarrow{p_2} & B \\
\end{array}
\]
(a) Prove that the categories:

- \( \text{Alg}_\Sigma \) for \( \Sigma \) a signature
- \( \text{DAut}_X \) for \( X \) a set of inputs
- \( \text{Cat} \)
- \( \text{Pos}(w) \) (w-cpos)
- \( \text{Rel} \)

all have products, by giving the detailed construction in each case and checking the required property.

Hint: For many concrete categories (recall the definition: 

(\( f \) is concrete iff there is a faithful functor 

\( \mathcal{U} : \mathcal{C} \rightarrow \text{Set} \)), the product \( A \times B \) in \( \mathcal{C} \) is such that \( \mathcal{U}(A \times B) = \mathcal{U}(A) \times \mathcal{U}(B) \), i.e. \( \mathcal{U} \) is constructed as the Cartesian product of the underlying sets. In such a case, the uniqueness of \( \langle f, g \rangle \) is guaranteed by its uniqueness as a function

\[ \langle f, g \rangle : \mathcal{U}(C) \rightarrow \mathcal{U}(A) \times \mathcal{U}(B) \]

Therefore, only the existence of \( \langle f, g \rangle \) needs to be proved.

(b) Prove that for \( \text{Pfn} \), the category of sets and partial functions, the cartesian product \( A \times B \) of sets is not a product in \( \text{Pfn} \). Prove, however, that \( \text{Pfn} \) has products, and give an explicit construction.
(c) Prove that if \( C \) has binary products, then the mapping \((A, B) \mapsto A \times B\) extends naturally to a functor \(-\times- : C^2 \to C\) by defining precisely how \(-\times-\) acts on arrows, where, by conventional notation, \(C^2\) is the category \(C \times C\).

(d) For any category \(C\) define the diagonal functor

\[
\delta_C : C \to C^2
\]

Prove that \(C\) has binary products iff \(\delta\) is a left adjoint. Conclude from this that \(A \times B, P_1, P_2\) are uniquely determined up to isomorphism.

(e) Let \(2 = \{0, 1\}\) be the two-element set, which is also the number 2 in the set-theoretic construction of the natural numbers (see STACS). Then the discrete poset \(2 = D(2) = (2, \leq_2)\) is a very simple category, which we can represent pictorially as:

\[
\begin{array}{c}
\circ 0 \\
0 \\
\circ 1 \\
1
\end{array}
\]

Prove that \(C^2\) and \([2 \to C]\) are isomorphic categories.
(f) Let \( \hat{S}_E : \mathcal{C} \to \mathcal{C} \) be the "diagonal functor" such that \( \hat{S}_E(A)(0) = \hat{S}_E(A)(1) = A \).

Prove that your isomorphism \( f \cong \mathcal{C} \xrightarrow{H} \mathcal{C} \) relates \( S \) and \( \hat{S}_E \) by the commutative diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{S} & \mathcal{C} \\
\downarrow & \cong & \downarrow H \\
\hat{S}_E & & \end{array}
\]

Conclude from this and Problem 2 that \( \mathcal{C} \) has binary products iff \( \hat{S}_E : \mathcal{C} \to \mathcal{C} \) is a left adjoint. Describe this adjointness in detail.

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**Problem 4**

**Coproducts**

Intuitively, coproducts allow us to "glue objects together" in a category. You may benefit from looking at Section 5.5 of STACS to see how, for sets, coproducts are exactly disjoint unions.

**Definition** A category \( \mathcal{C} \) has (binary) coproducts iff for each \( A, B \in \mathcal{C} \) there is an object denoted \( A \oplus B \in \mathcal{C} \) and called their coproduct, and arrows \( i_1 : A \to A \oplus B, \ i_2 : B \to A \oplus B \), called the
injections from $A$ and $B$, such that for any object $C \in \mathcal{C}$ and any pair of arrows $f : A \rightarrow C$, $g : B \rightarrow C$ there exists a unique arrow $\exists f, g : A \oplus B \rightarrow C$ such that

\[
\begin{array}{c}
\xymatrix{ & C \ar[dl]_f \ar[dr]^g & \\
A \ar[rr]^{i_1} & & A \oplus B \ar[rr]^{i_2} & & B}
\end{array}
\]

(a) Prove that the categories:
- $\text{Vect}_{\mathbb{R}}$ (finite-dimensional vector spaces over the reals $\mathbb{R}$ and linear functions)
- $\text{Pos}$
- $\text{Cat}$
- $\text{Graph}$
- $\text{Pfn}$
- $\text{Rel}$

have coproducts.

(b) Prove that products and coproducts are dual concepts, i.e., $(A \times B, p_1, p_2)$ is a product in $\mathcal{C}$ iff $(A \times B, p_1 : A \rightarrow A \times B, p_2 : B \rightarrow A \times B)$ is a coproduct in $\mathcal{C}^{\text{op}}$

Use this duality to prove that:

(b.1) If $\mathcal{C}$ has binary coproducts, there is a functor $- \oplus - : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ extending $\oplus$ to arrows

- $\oplus$ : $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ extending $\oplus$ to arrows
(6.2) $\mathcal{C}$ has binary coproducts iff

$\eta: \mathcal{C} \to \mathcal{C}^2$ is a right adjoint.

(6.3) $\mathcal{C}$ has binary coproducts iff

$\eta': \mathcal{C} \to [2 \to \mathcal{C}]$ is a right adjoint.

Describe this adjointness in detail.

**Problem 5.** Let $\mathcal{C}$ be a category and $\equiv \subseteq A_\mathcal{C} \times A_\mathcal{C}$ an equivalence relation on its set of arrows. Call $\equiv$ a congruence iff:

1. $f \equiv g \implies s(f) = s(g) \land t(f) = t(g)$

2. If $f \equiv f'$, $g \equiv g'$, and $t(f') = s(g)$,
   then $f; g = f' ; g'$

(a) Define a category $\mathcal{C}/\equiv$, called the quotient of $\mathcal{C}$ modulo $\equiv$, such that $0_{\mathcal{C}/\equiv} = 0_{\mathcal{C}}$, and $A_{\mathcal{C}/\equiv} = A_{\mathcal{C}/\equiv}$, and define a full functor $Q_\equiv: \mathcal{C} \to \mathcal{C}/\equiv$.

(b) Prove that if $F: \mathcal{C} \to \mathcal{D}$ is a functor such that $\forall (f; g) \equiv \equiv$, $F(f) = F(g)$, then there exists a unique functor $\bar{F}: \mathcal{C}/\equiv \to \mathcal{D}$ such that:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow{Q_\equiv} & & \downarrow{\bar{F}} \\
\mathcal{C}/\equiv & \xrightarrow{\bar{F}} & \mathcal{D}
\end{array}
$$
(c) Prove that if \( R \subseteq A \times A \) is a binary relation such that:

\[
(1) \quad f \text{ R } g \Rightarrow s(f) = t(g) \land t(f) = t(g)
\]

then there is a smallest congruence \( \overline{R} \subseteq A \times A \) such that \( R \subseteq \overline{R} \).

(d) Prove that if \( F : \mathcal{C} \rightarrow \mathcal{D} \) is a functor such that

\[\forall (f, g) \in R, \quad F(f) = F(g),\]

then there exists a unique functor \( \overline{F} : \mathcal{C}/\overline{R} \rightarrow \mathcal{D} \) such that

\[
\begin{array}{rcl}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow & & \downarrow \overline{F} \\
\mathcal{C}/\overline{R} & \xrightarrow{\overline{F}} & \mathcal{D}
\end{array}
\]

Problem 6. Since \( \text{Pos} \hookrightarrow \text{Cat} \) is a full subcategory, we can ask the obvious question of what inclusion, we can ask the obvious question of what inclusion, we can ask the obvious question of what inclusion, we can ask the obvious question of what

products and coproducts mean when the category \( \mathcal{C} \) is a poset \( P = (P, \leq) \). Characterize the existence of products (resp. coproducts) in a poset \( P = (P, \leq) \) in terms of familiar notions in the theory of partially ordered sets.