# Announcements

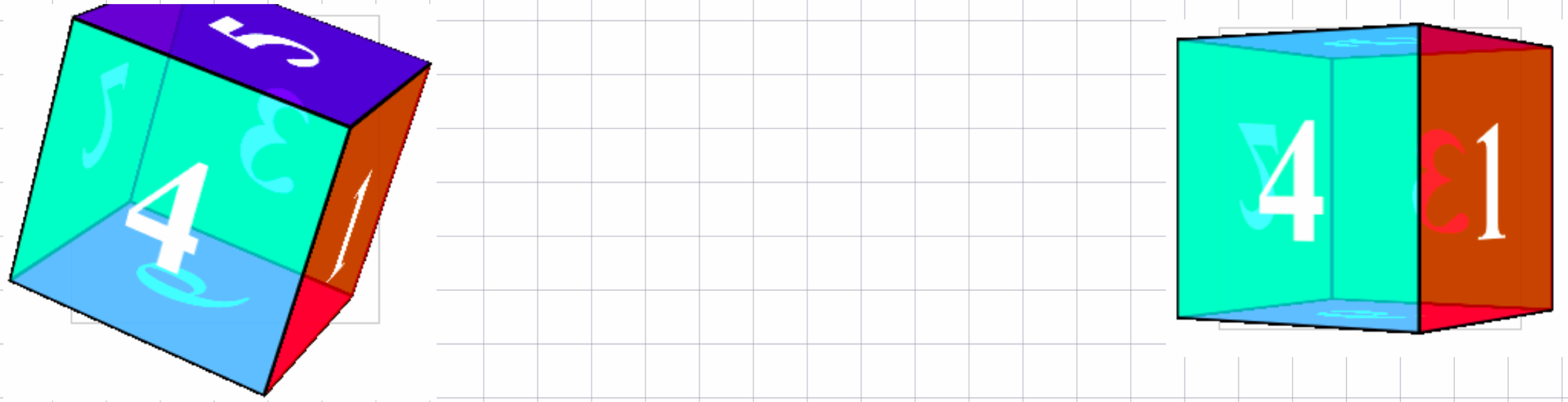
- Work in groups - groups of 2 for MPs, groups of 2-4 for the final project.
- Final project webpage: coming soon.
- MP1 is due on Sep 15, 11:59pm

# 2D and 3D Matrices: Linear Tranformations

http://www.html5rocks.com/en/tutorials/webgl/webgl_transforms/webgl/webgl-2d-geometry-matrix-transform.html

http://www.senocular.com/flash/tutorials/transformmatrix/examples/3dpicturebox.html

# 2D Linear Transformations: Compositions



# 2D Linear Transformations: Compositions

1. Order of multiplication matters?

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}$$

2. Which one gets applied first?

$$M_3 = M_2 \cdot M_1$$

http://math.mercyhurst.edu/~lwilliams/Applets/LinearTransformations.html

# 2D Linear Transformations: Inverses

**Def:**

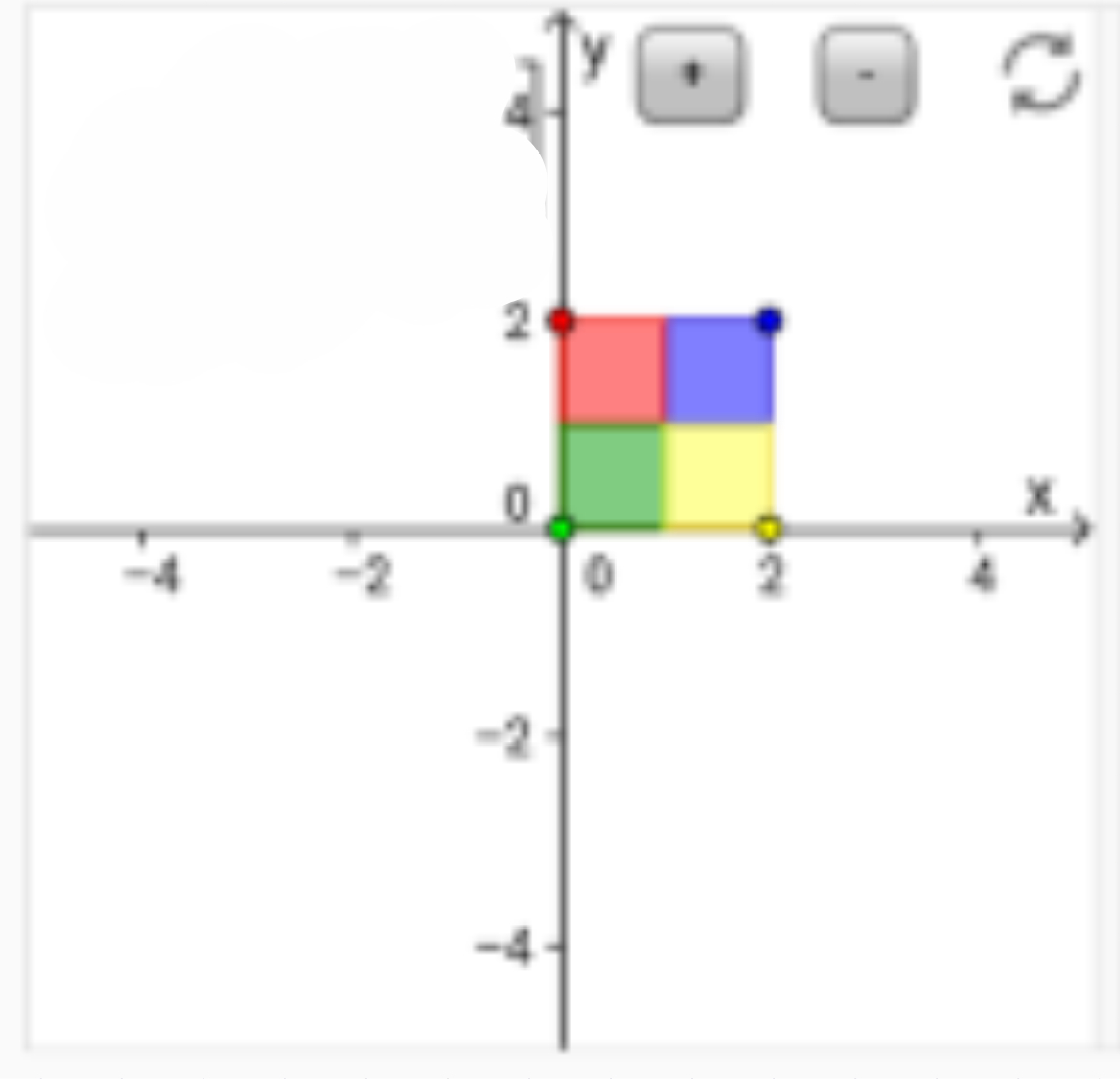
**Stretch:** $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} & \\ & \end{bmatrix}$

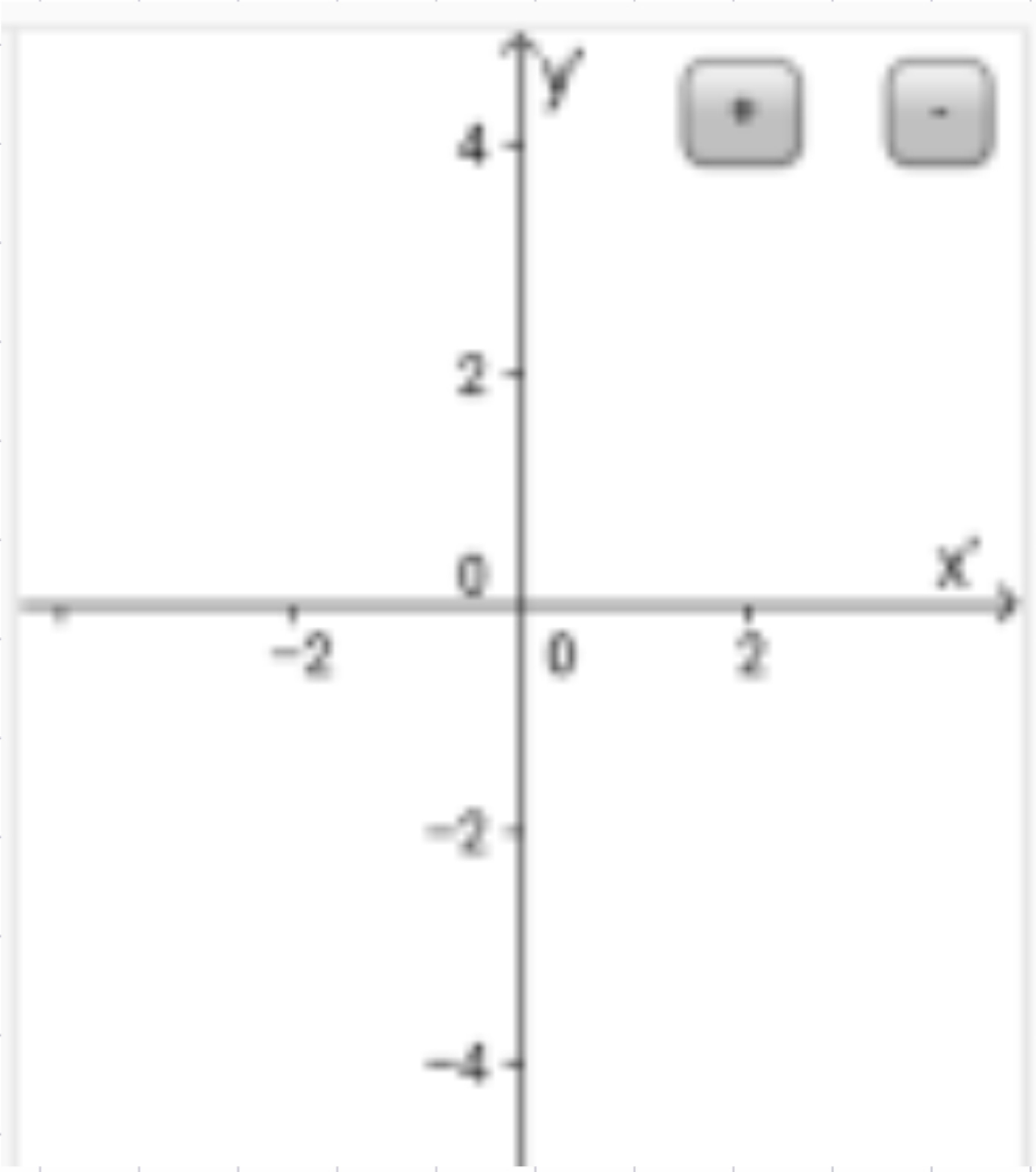
**Shear:** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} & \\ & \end{bmatrix}$

**Rotation:** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1}$

**Composition:** $(M_4 \cdot M_3 \cdot M_2 \cdot M_1)^{-1} =$

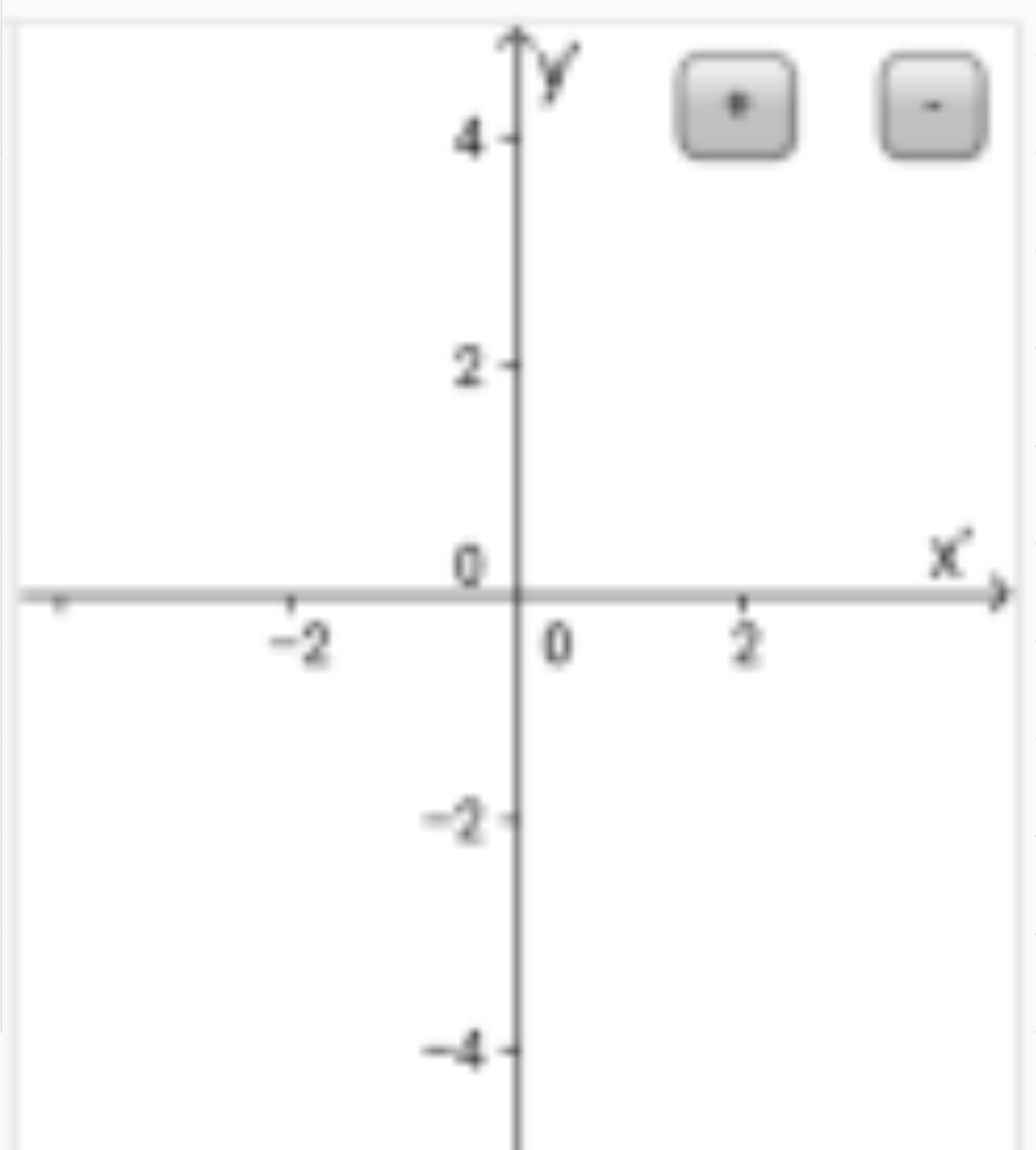
# 2D Linear Transformations: Review





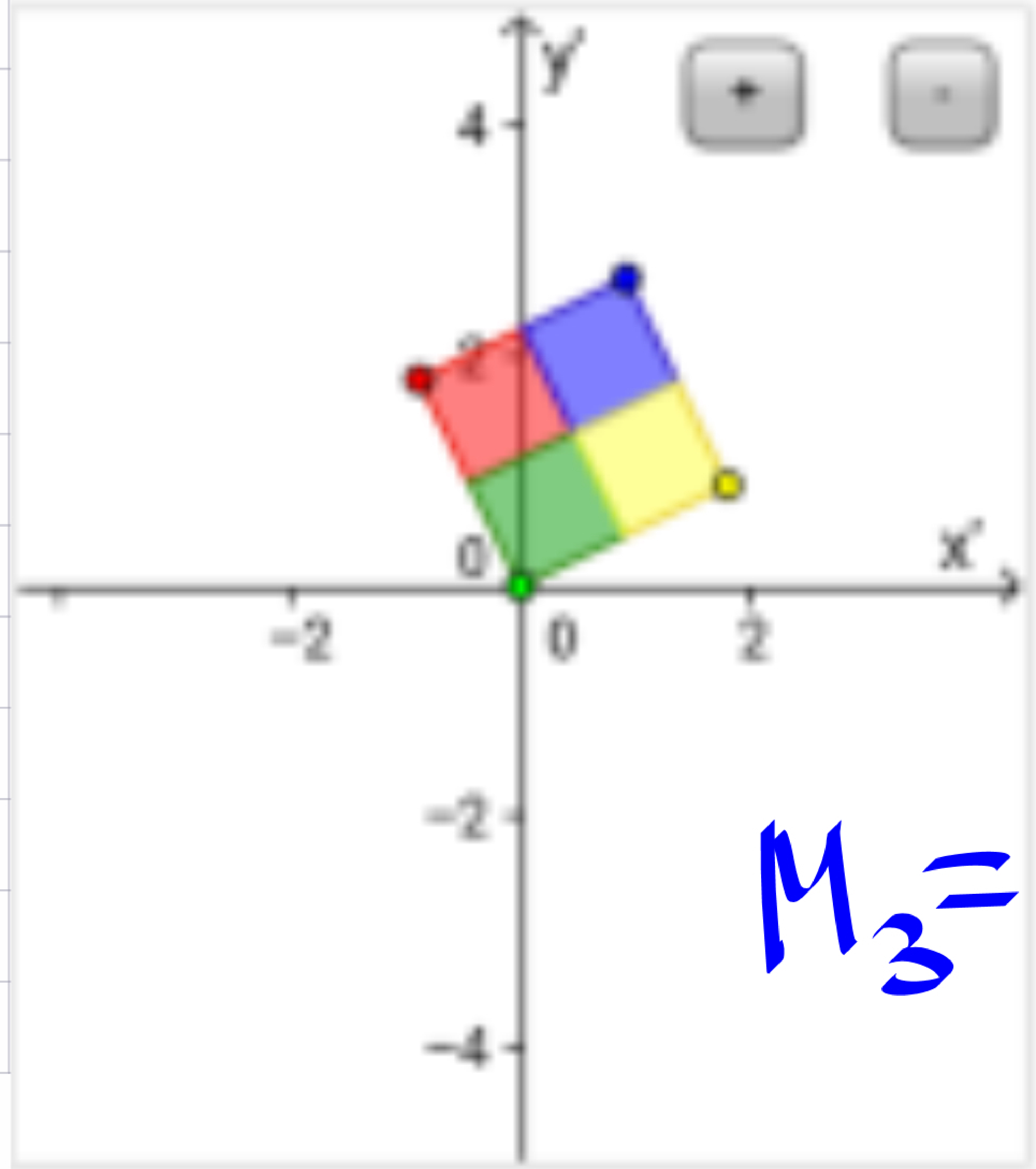
Draw:

$$M_1 = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}$$



Draw:

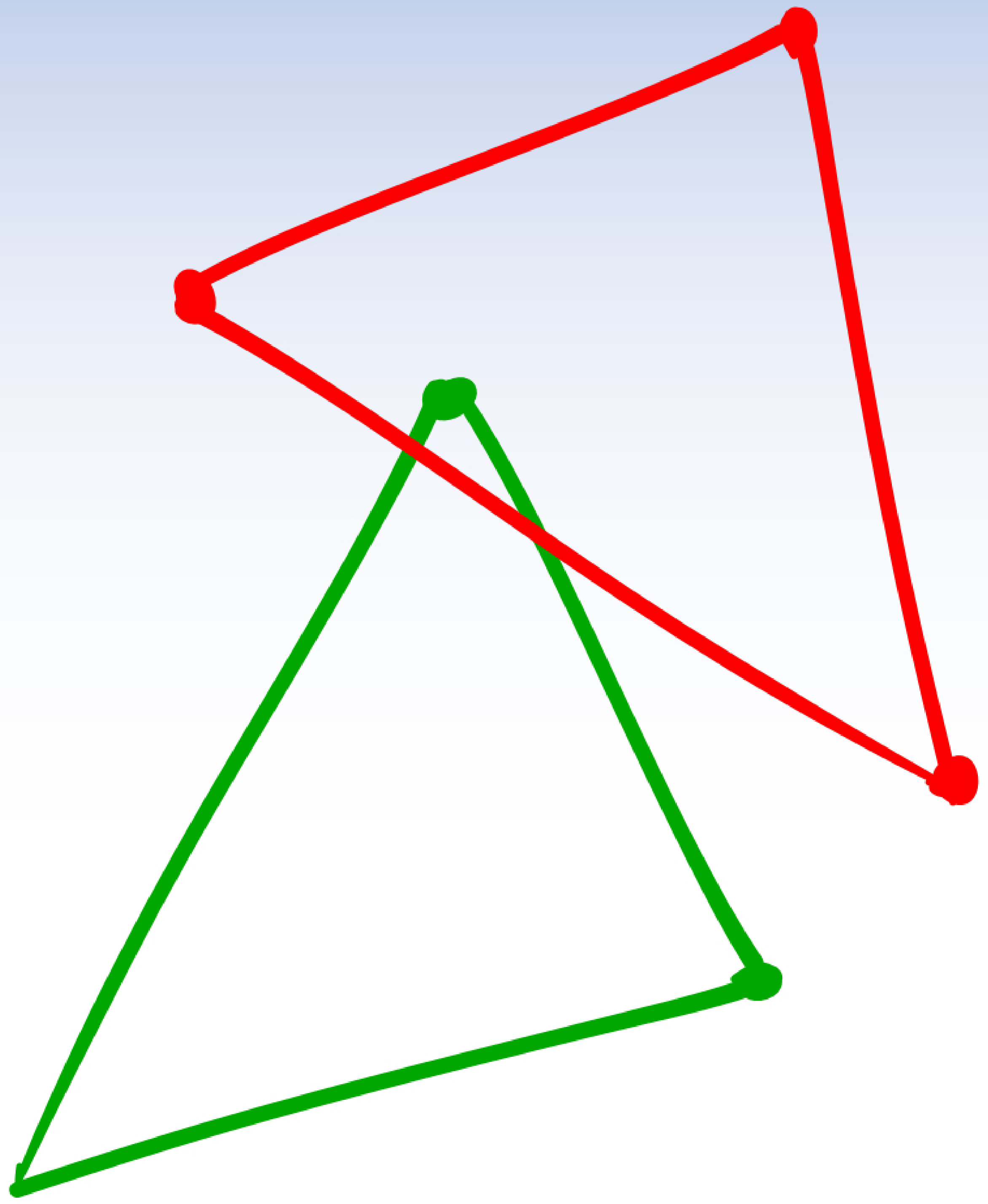
$$M_2 = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$



Find rotation:

$$M_3 = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

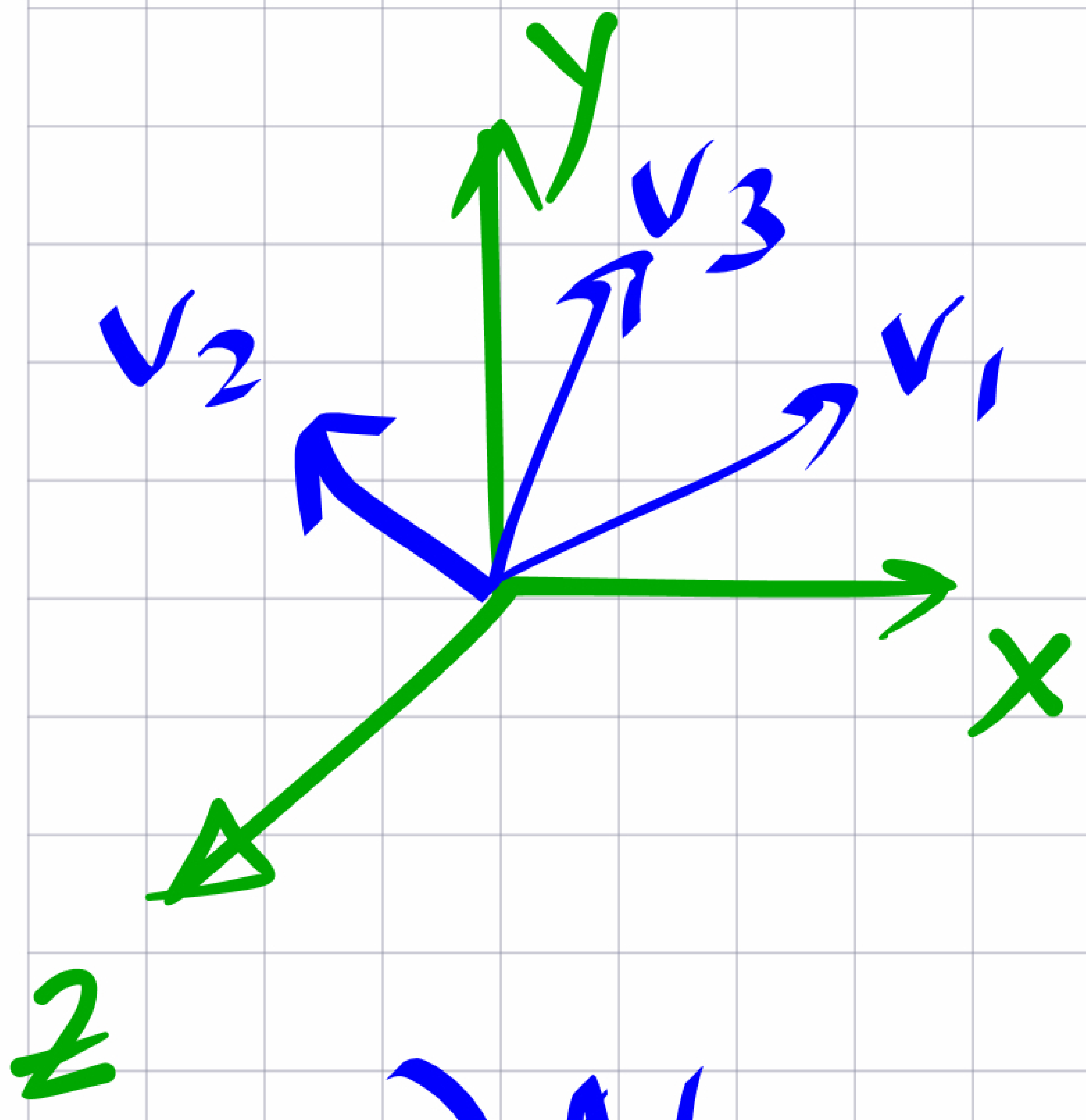
# Rigid Body Transformations



DOFs

| | 2D | 3D |
|---|---|---|
| 1) Easy: translations | 2 | 3 |
| 2) More difficult: rotations | 1 | 3 |
| 3) Most difficult: rotation + translation | 3 | 6 |

# Rotations: 3D



$$M = \left[\begin{array}{ccc} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ \underbrace{m_{31}}_{v_1} & \underbrace{m_{32}}_{v_2} & \underbrace{m_{33}}_{v_3} \end{array}\right] \quad 9\ \text{DOFs}$$

1) No scaling

$$\|v_1\| = 1$$
$$\|v_2\| = 1$$
$$\|v_3\| = 1$$

$$\boxed{-3}$$

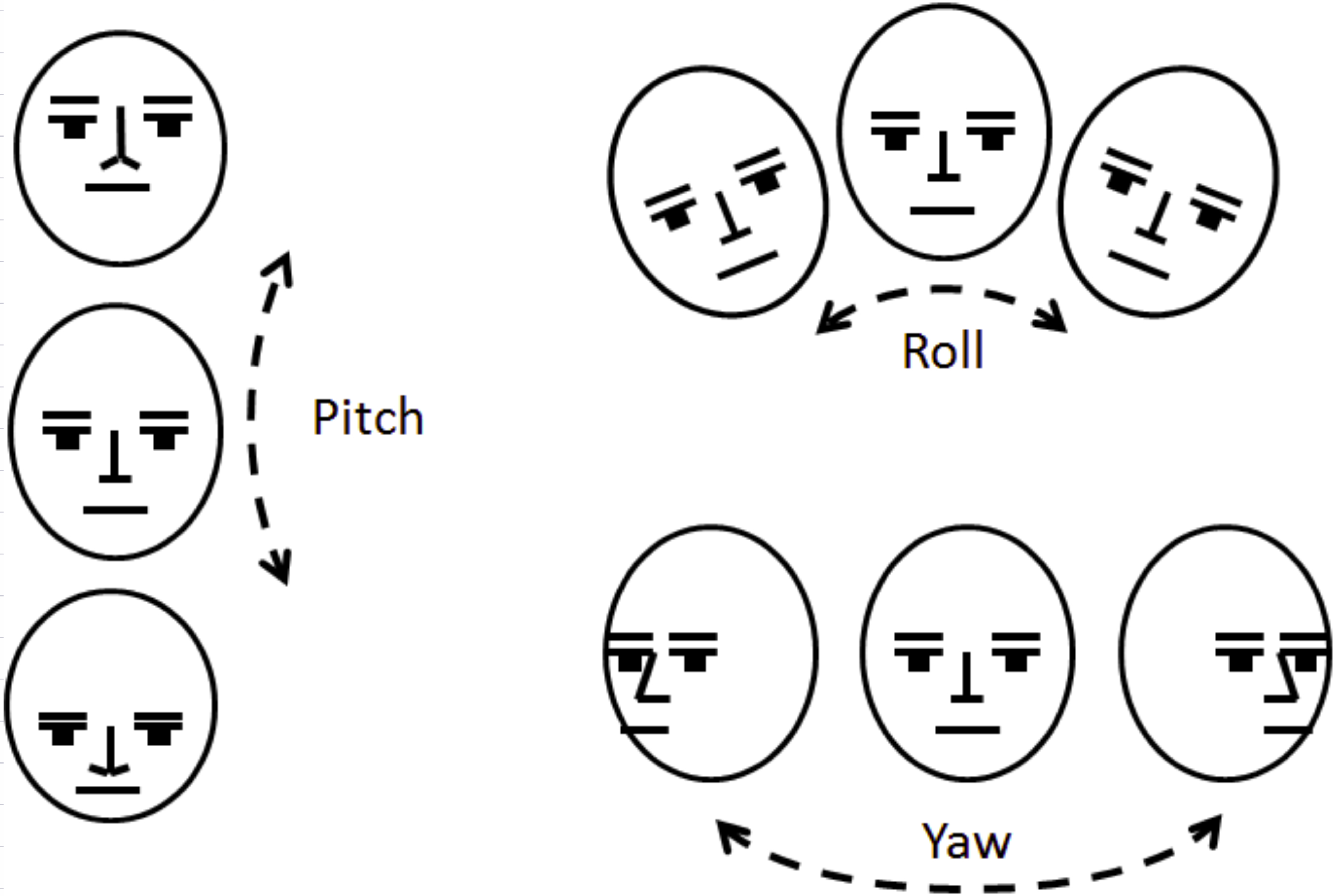
2) No shearing

$$v_1 \cdot v_2 = 0$$
$$v_2 \cdot v_3 = 0$$
$$v_1 \cdot v_3 = 0$$

$$\boxed{-3}$$

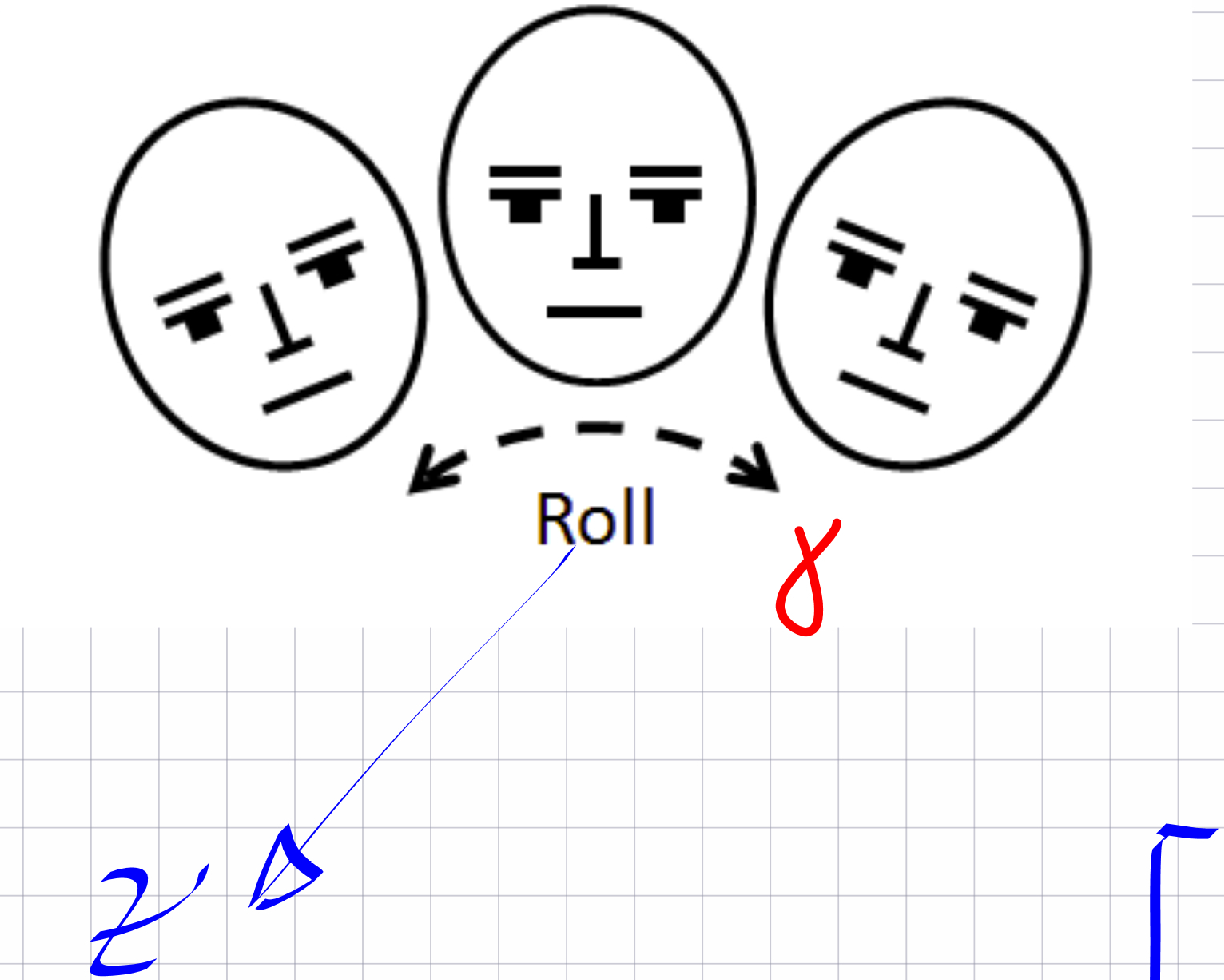
3) $\det(M) = \overset{+}{\cancel{-}}1$

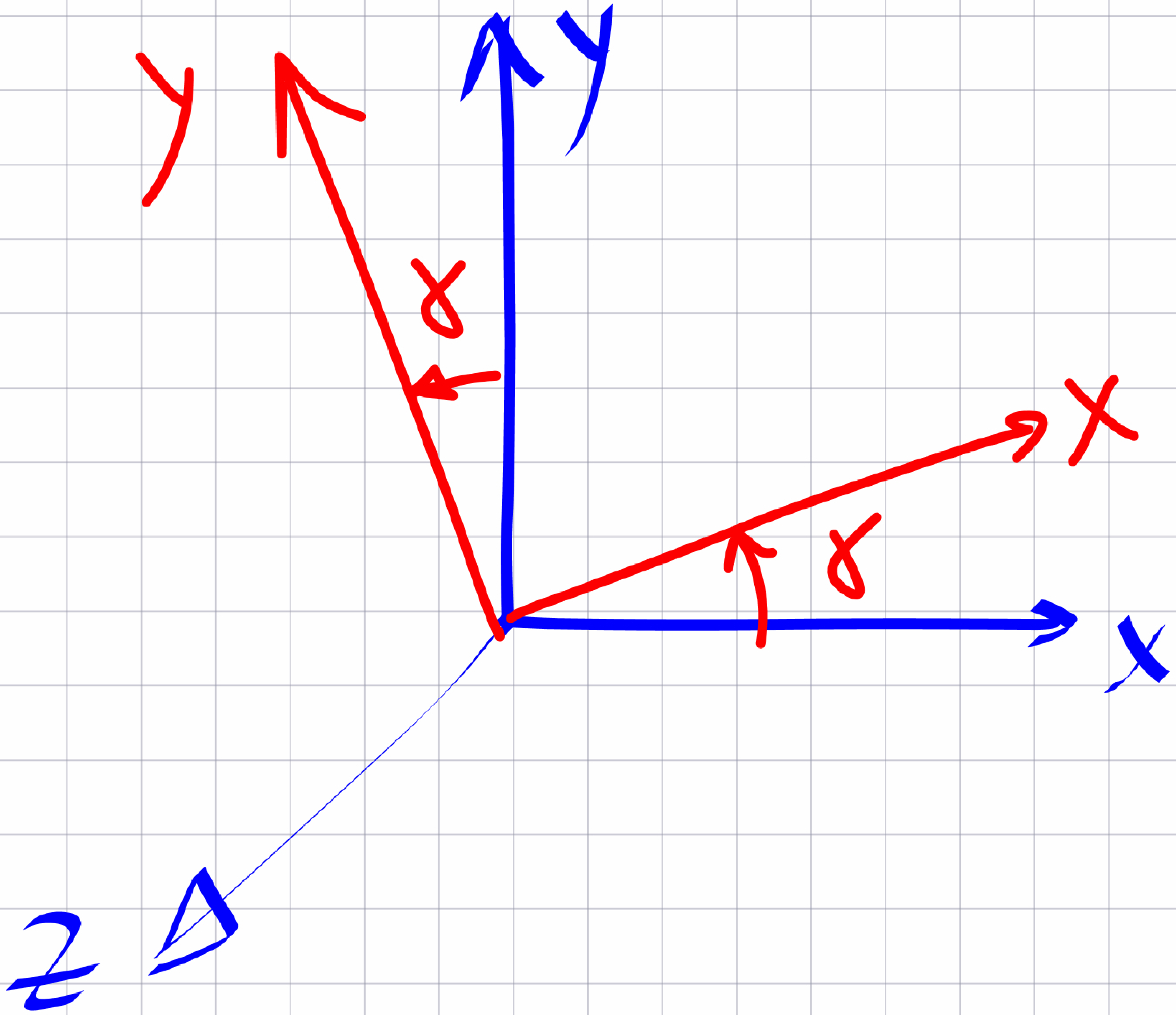
# 3D Canonical Rotations: Yaw, Pitch, Roll



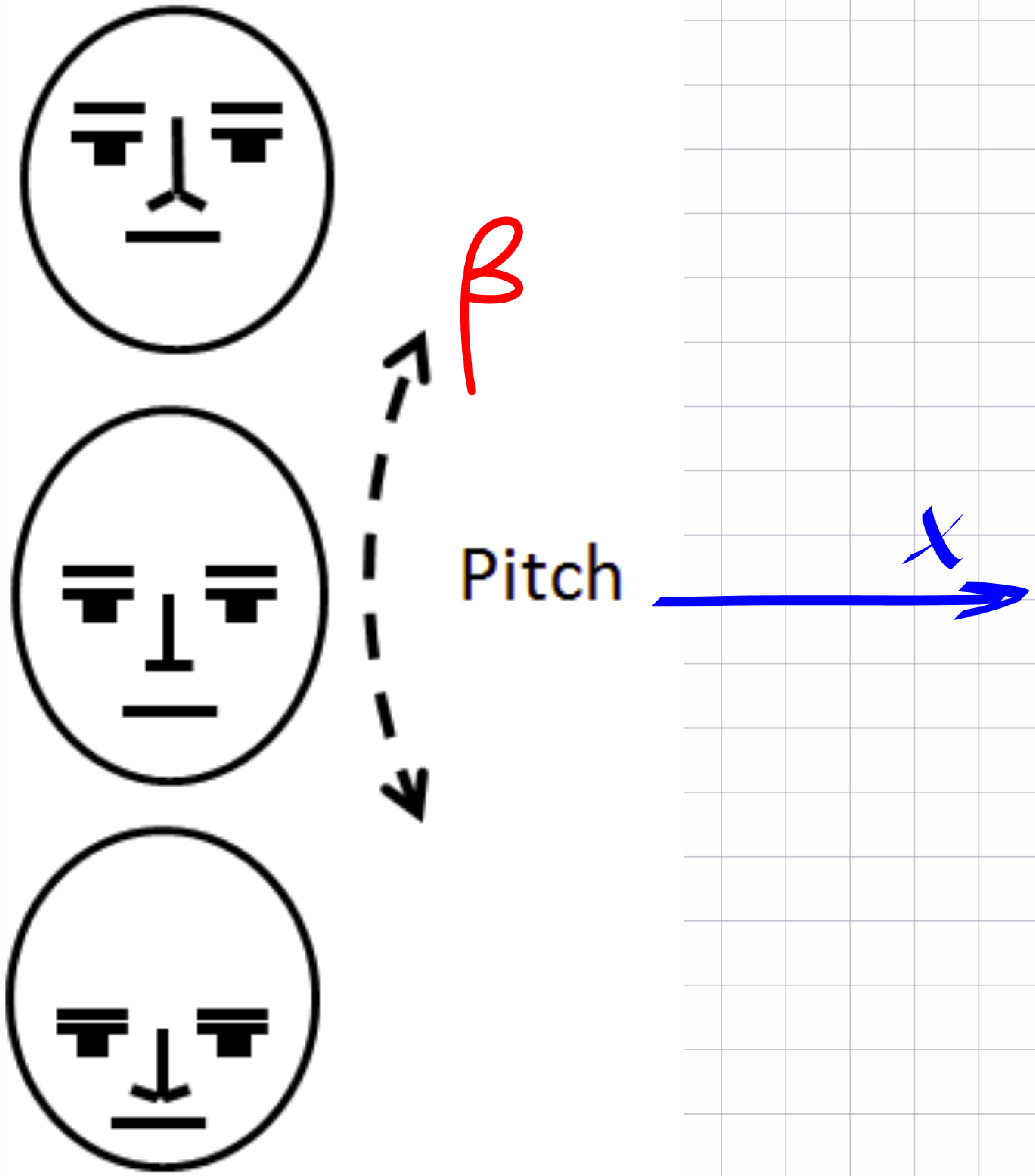
Also called Euler angles

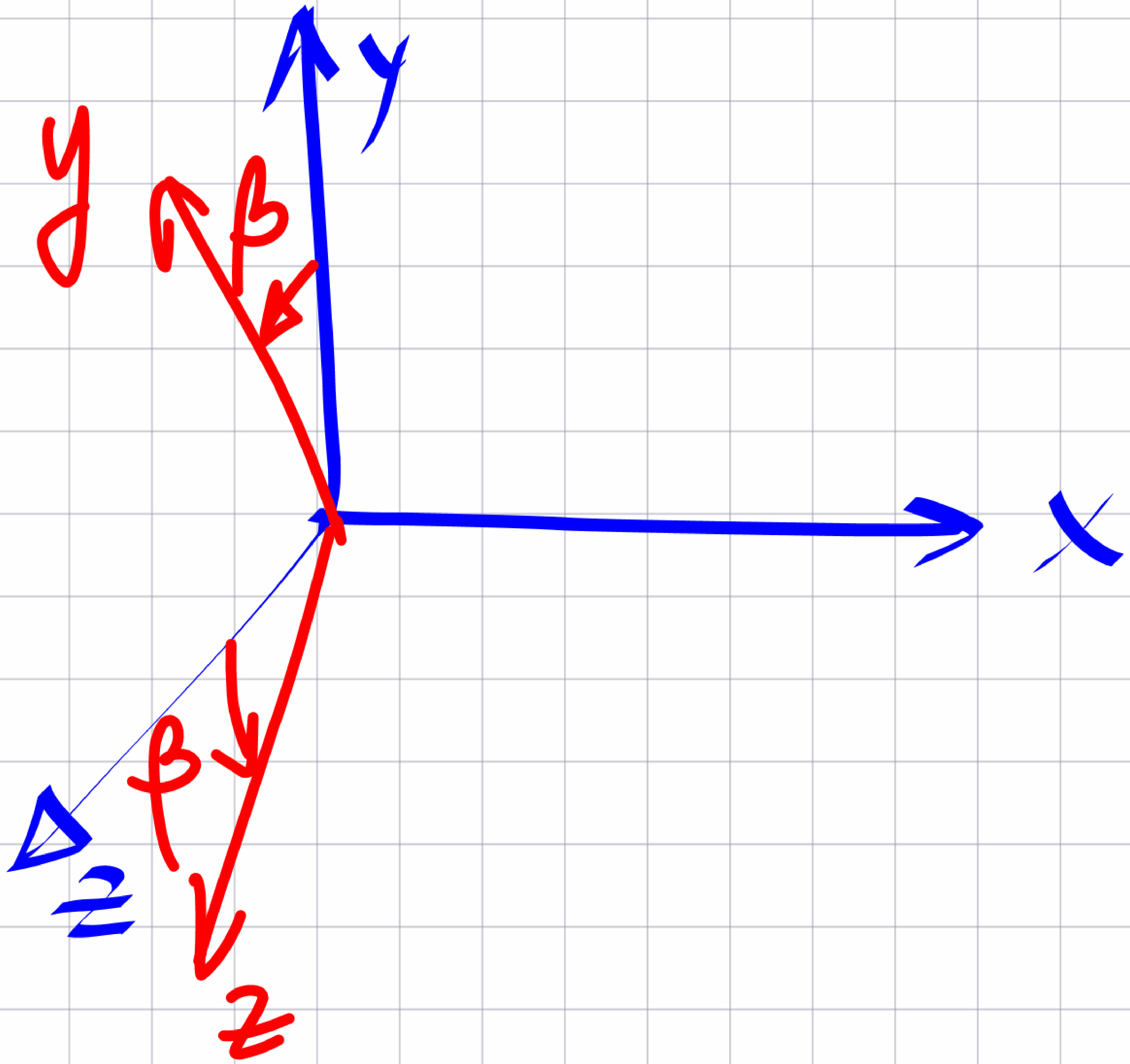
# 3D Canonical Rotations: Roll



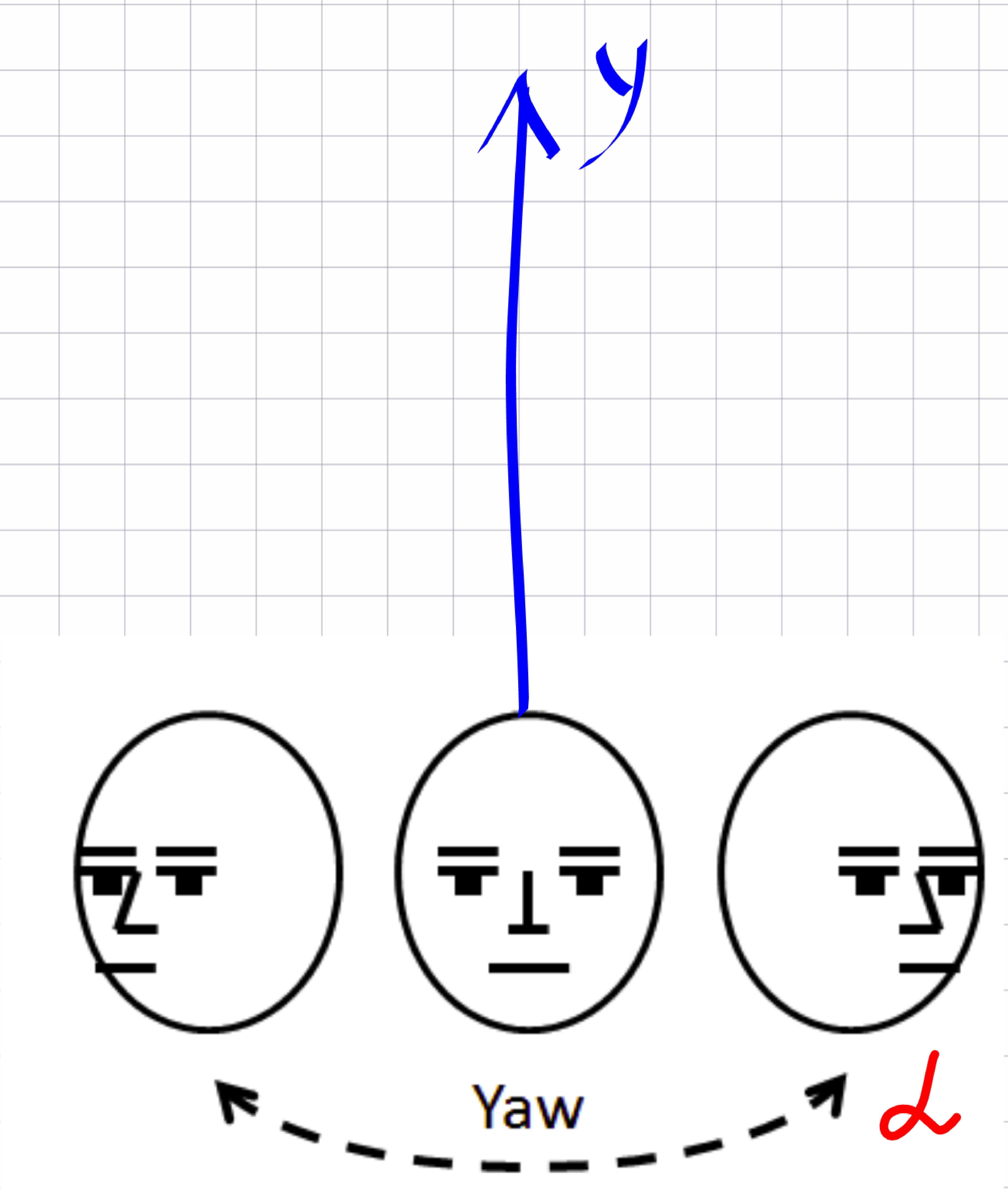


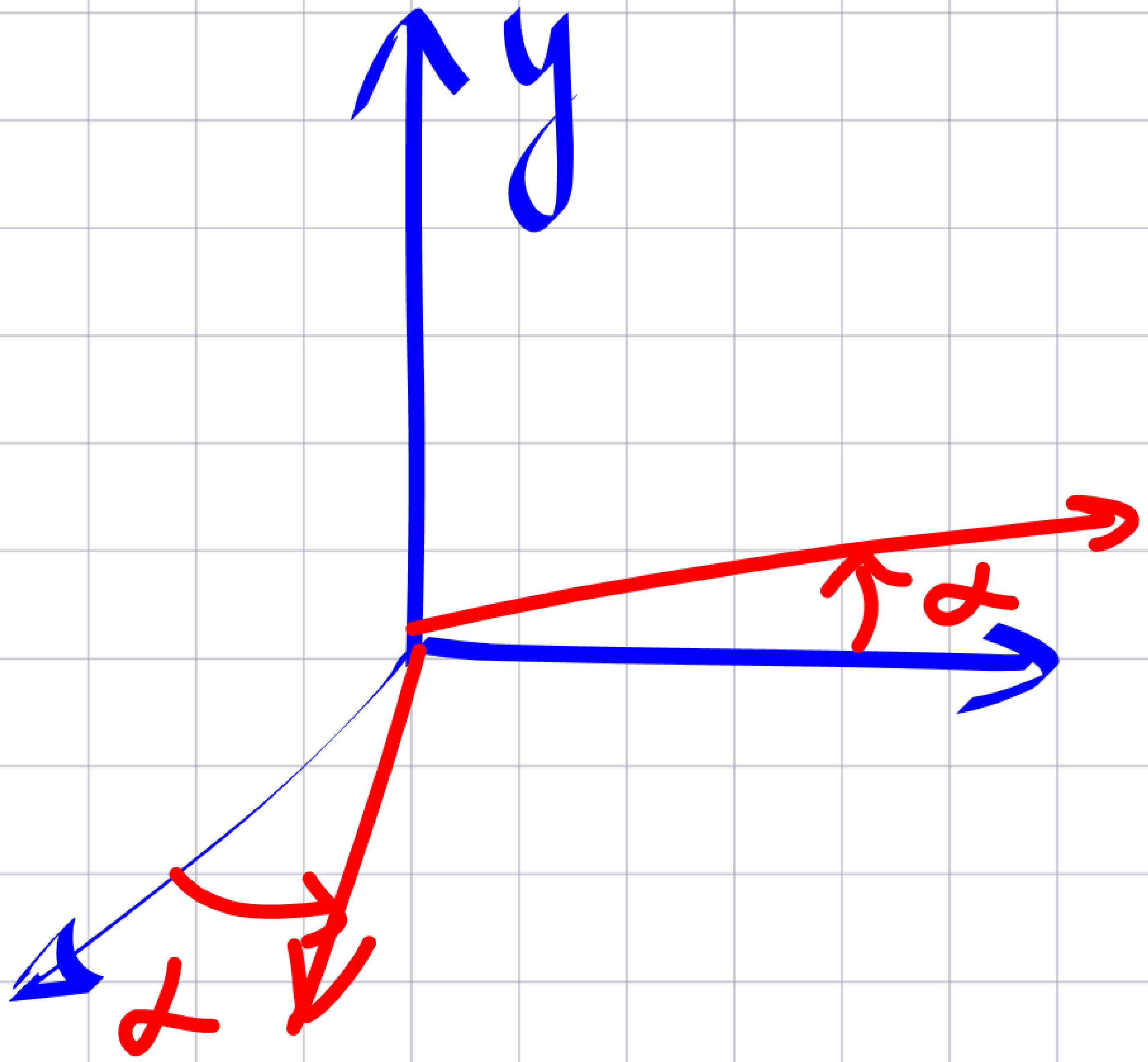
# 3D Canonical Rotations: Pitch



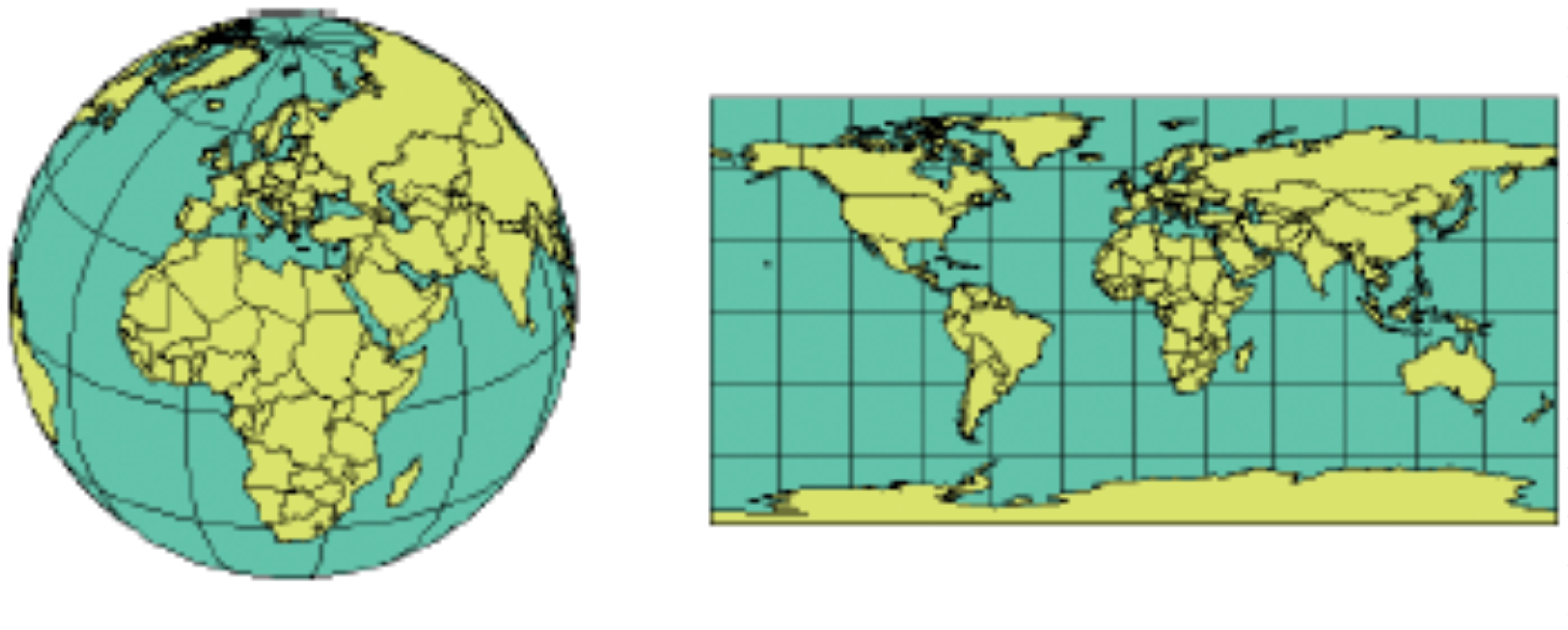


# 3D Canonical Rotations: Yaw





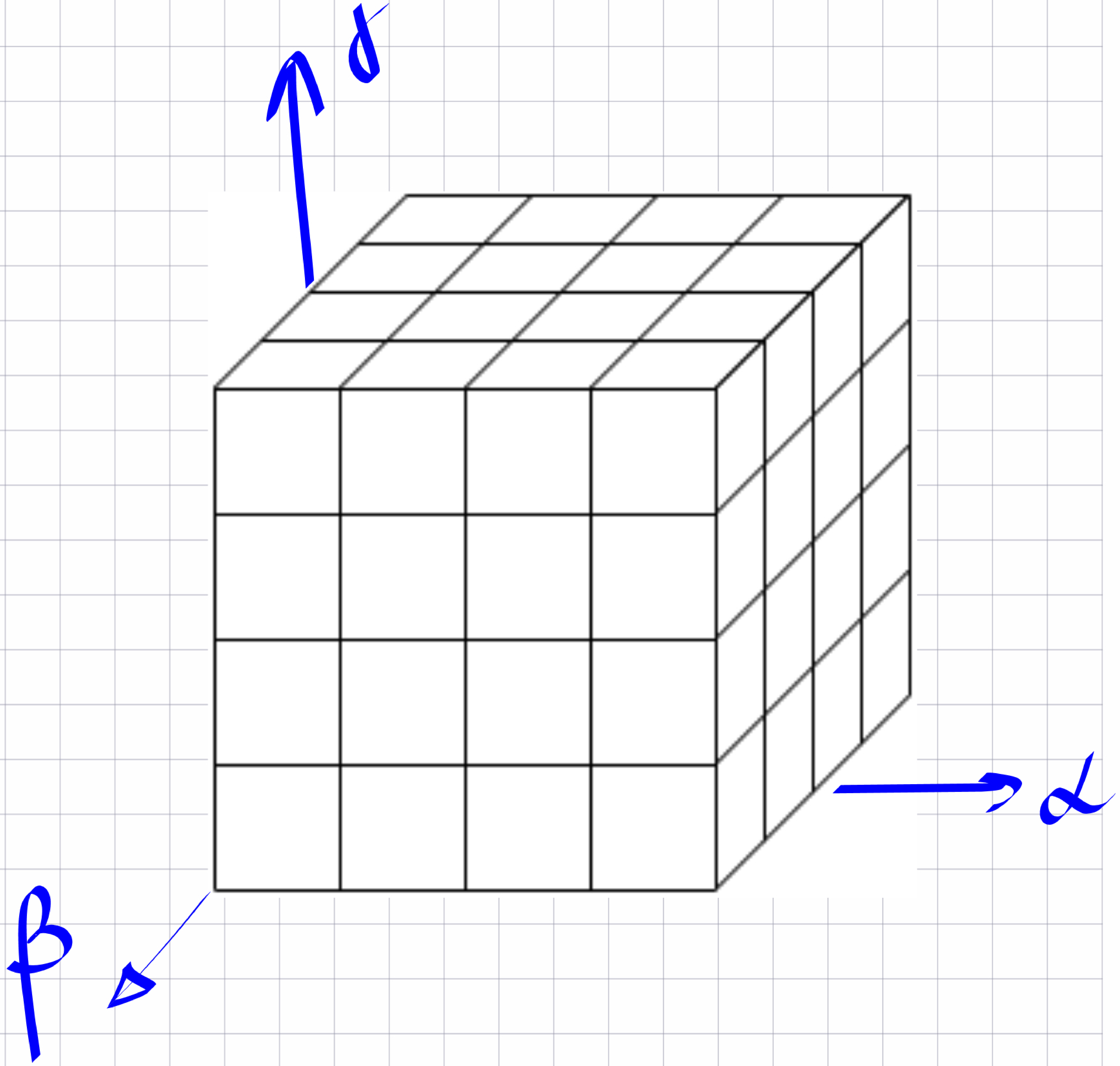
# Parameterizing a 2D Sphere



1. Singularities

2. Shortest distances

# Parameterizing Rotations: Yaw, Pitch, Roll



1. Singularities

2. Shortest distances

# Parameterizing Rotations: Yaw, Pitch, Roll

1. Commutativity:

$$\overset{\text{yaw}}{R_y}\left(\frac{\pi}{2}\right) \cdot \overset{\text{roll}}{R_z}\left(\frac{\pi}{2}\right)$$

$$\underset{\text{roll}}{R_z}\left(\frac{\pi}{2}\right) \cdot \underset{\text{yaw}}{R_y}\left(\frac{\pi}{2}\right)$$

2. Singularity (Gimbal Lock):

$$R_y(\alpha) \cdot R_x\left(\frac{\pi}{2}\right) \cdot R_z(\gamma) =$$

$$\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

# Representing Rotations: Euler's Rotation Theorem

Theorem:

1. Axis- Angle Representation

2. Exponential  Coordinates

# Representing Rotations: Unit Quaternions

$q = (a, b, c, d) \in \mathbb{R}^4$ , $a^2 + b^2 + c^2 + d^2 = 1$

Set of all $q$ is a

https://www.wolframalpha.com/input/?i=quaternion%3A+0%2B2i-j-3k&lk=3

# Unit Quaternions: Examples

$(1, 0, 0, 0)$

$(0, 1, 0, 0)$

$(0, 0, 1, 0)$

$(0, 0, 0, 1)$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)$

$\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$

$\left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)$

# Unit Quaternions: Inverses and Duplicates

https://www.wolframalpha.com/input/?i=quaternion%3A+0%2B2i-j-3k&lk=3