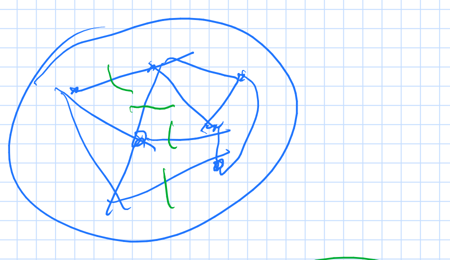
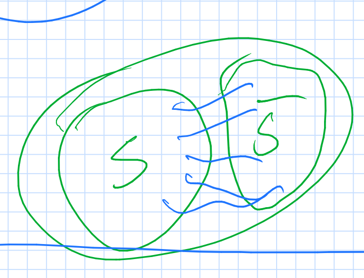
4/1/21

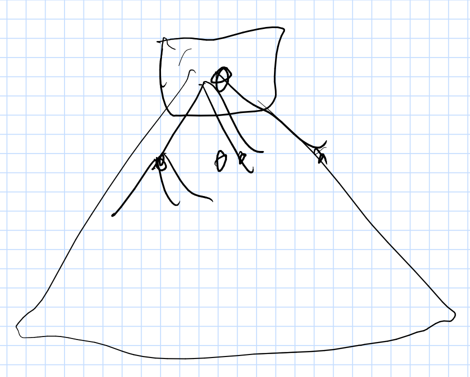
min cut



min #
max-cut



Galton-Watson process



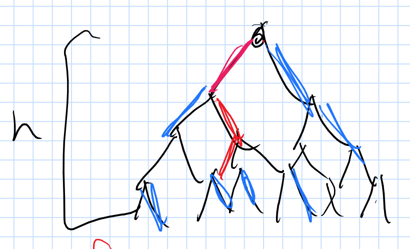
$X \in \{0,1,2,\dots\}$

$E[X] > 1$

$E[X] < 1 \Rightarrow$ The tree is finite with prob 1

$E[X] = 1 \Rightarrow$ the family name is finite.

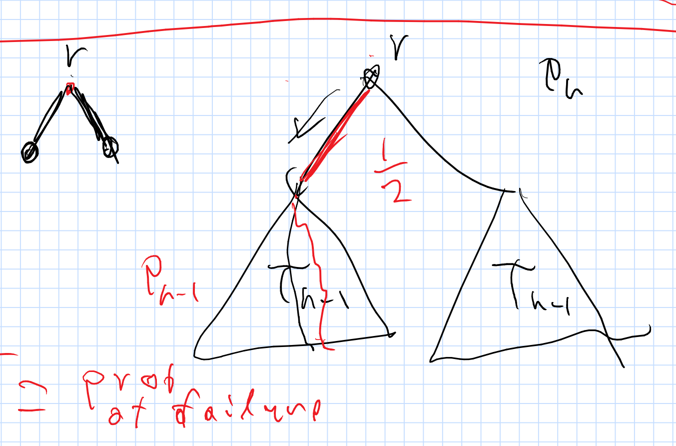
Coloring binary trees



T_h balanced binary tree
2^h leafs

$P_h = P$[T_h has a red path from the root to one of the leafs]

$P_1 = \frac{3}{4}$



$\left(1-\frac{P_{n-1}}{2}\right)^2 =$ prob of failure

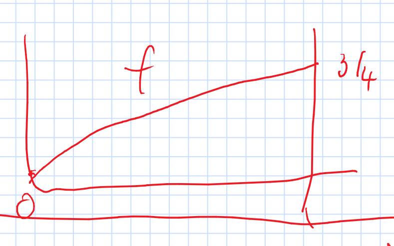
$P_n = 1-\left(1-\frac{P_{n-1}}{2}\right)^2$

$= \left(1-\left(1-\frac{P_{n-1}}{2}\right)\right)\left(1+1-\frac{P_n}{2}\right)$

$= \frac{P_{n-1}}{2}\left(2+\frac{P_{n-1}}{2}\right) = P_{n-1} - \left(\frac{P_{n-1}}{2}\right)^2$

$f(x) = x - \frac{x^2}{4}$

$P_n = f(P_{n-1})$



Lemma $P_n \geq \frac{1}{n+1}$ $\qquad P_n = O\left(\frac{1}{n}\right)$

Proof $P_1 = \frac{3}{4} \geq \frac{1}{2}$

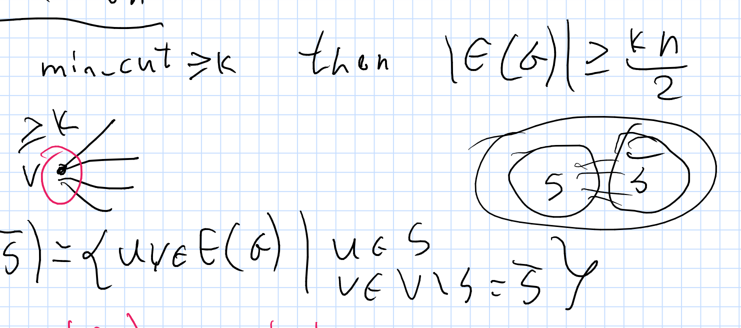
$P_n = f(P_{n-1}) \geq f\left(\frac{1}{n}\right) \geq \frac{1}{n+1}$ ∎

Min cut

$K = \#$ edges in min cut

Observation

If min-cut $\geq K$ then $|E(G)| \geq \frac{Kn}{2}$

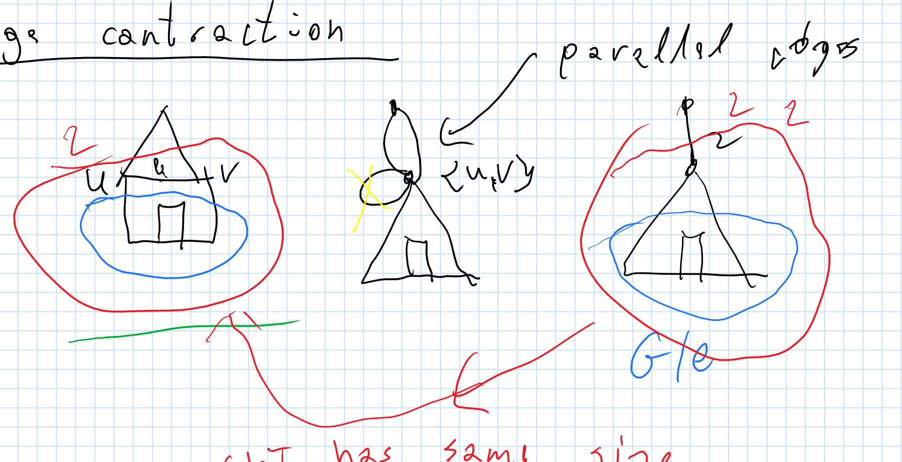


$(S,\bar{S}) = \{uv \in E(G) \mid u \in S, v \in V \setminus S = \bar{S}\}$

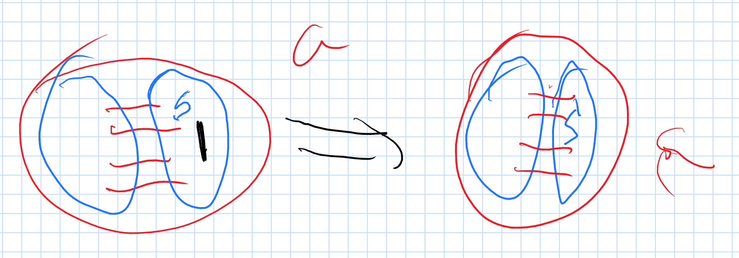
$Kn \leq \sum_{v \in V(G)} d(G) = 2|E|$

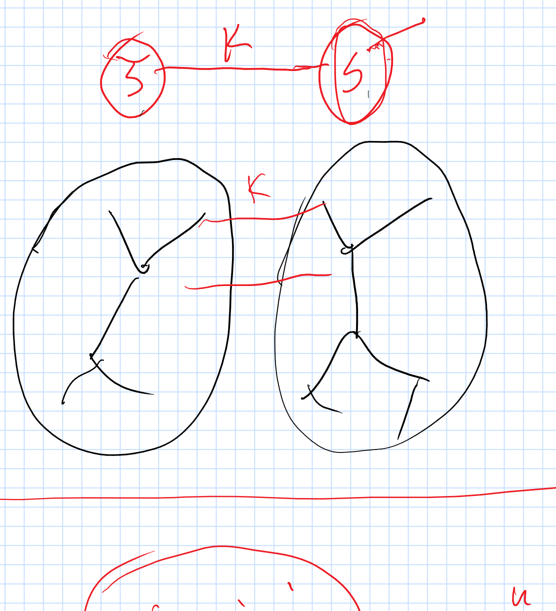
$\Rightarrow |E| \geq \frac{Kn}{2}$.

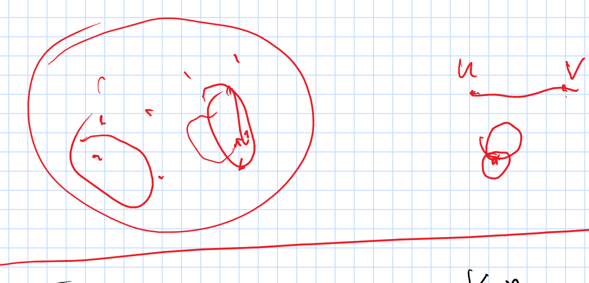
Edge contraction

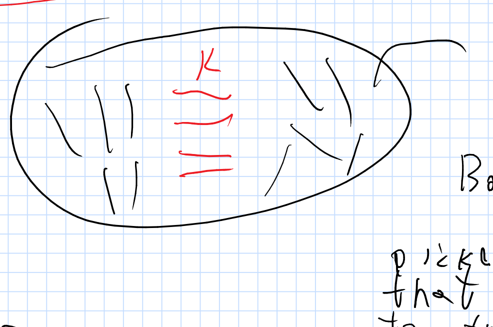


$2^n - 2$









$\frac{Kn}{2}$

Bad event

picked edge that belongs to the min cut.

P[bad event first iteration]

$= \frac{K}{|E(G)|} \leq \frac{K}{nK/2} = \frac{2}{n}$. $\quad |E| \geq \frac{Kn}{2}$

P[first iteration succeeds] $= 1 -$ bad prob $\geq 1 - \frac{2}{n} = \frac{n-2}{n}$

$n \Rightarrow n-1$ vertices $\Rightarrow n-2 \qquad O(n)$

$n \Rightarrow n-1 \Rightarrow n-2 \Rightarrow \dots \Rightarrow 2$ vertice

$\frac{n-2}{n} \quad \frac{n-3}{n-1} \quad \frac{n-4}{n-2} \qquad \frac{1}{3}$

P[alg succeeds]

$\geq \frac{n-2}{n} \frac{n-3}{n-1} \frac{n-4}{n-2} \frac{n-5}{n-3} \dots \frac{3}{5} \frac{2}{4} \frac{1}{3}$

$= \frac{2}{n(n-1)} \geq \frac{2}{n^2} \qquad O(n^2) \Leftarrow$

$p = \frac{2}{n^2} \qquad \frac{1}{p}$

$(1-p)^{1/p} \leq (\exp(-p))^{1/p} = e^{p \cdot (-1/p)} = \frac{1}{e} \leq \frac{1}{2}$

$1 - x \leq \exp(-x) \qquad x \in [0,1]$

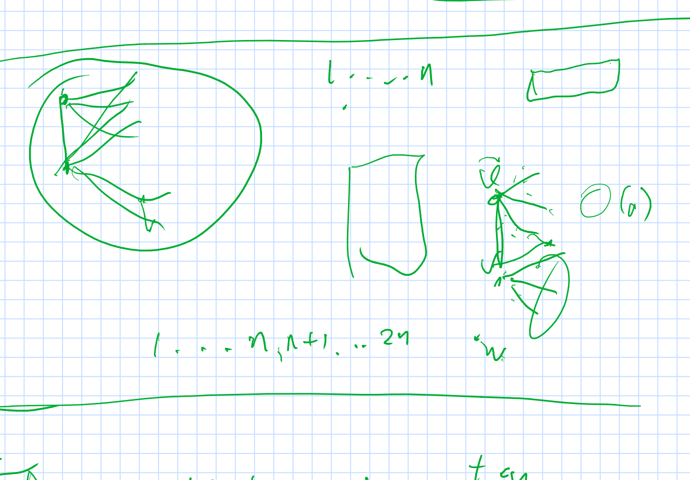
$O(n^2)$ times $\Rightarrow$ prob of success $\geq \frac{1}{2}$.

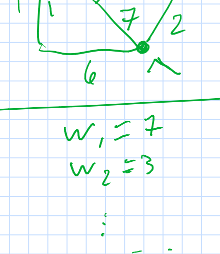
$(1-p)^m \leq \frac{1}{n^{10}} \qquad \Rightarrow \geq 1 - \frac{1}{n^{10}}$

$m = \frac{10 \ln n}{p}$

$(1-p)^m \leq \exp(-pm) = \exp\left(-p\frac{10 \ln n}{p}\right) = \frac{1}{n^{10}}$

$O(n^2 \log n) \; O(n^2) \qquad O(n^4 \log n)$





$w_1, w_2 \dots w_t \qquad t \leq n$

$w_1 = 7$
$w_2 = 3$
$\vdots$
$w_t = \dots$

$P[X = i] = \frac{w_i}{\sum w_j}$

$O(t)$

$\dots$ $O(1)$