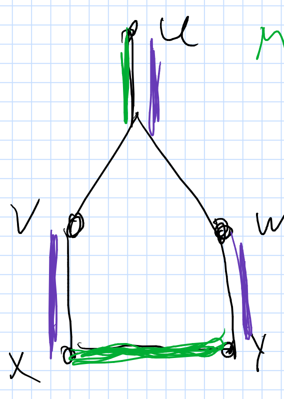
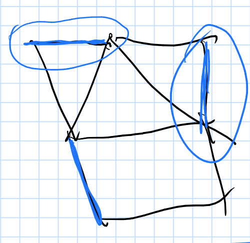
3/25/21

Matchings



Maximum cardinality (size) matching

Maximal matching is a matching that can not be made bigger by adding an edge.



$O(m)$ time

n: # vertices
m: edges

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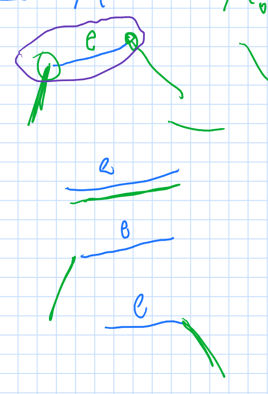
maximum matching
= largest matching among all matchings in the graph.

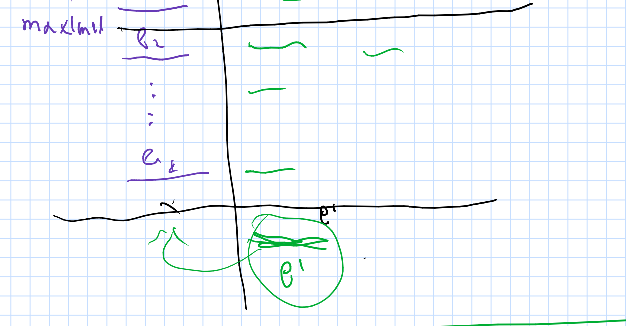
	bipartite	general
unw	✓	✓
weighted	✓	◯ ←

Lemma
Let M be a maximal matching
Let M_{opt} be a maximum matching.

$$|M| \geq \frac{|M_{opt}|}{2}$$

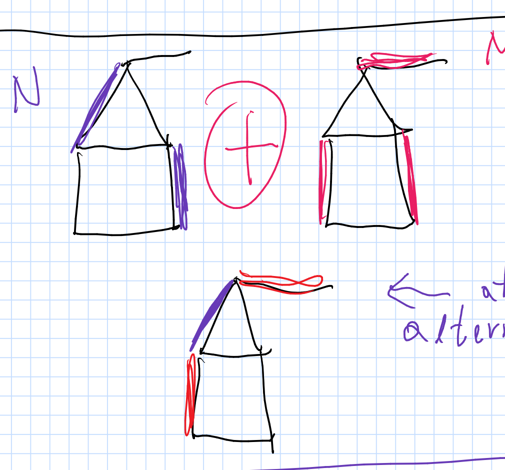
Proof M M_{opt}



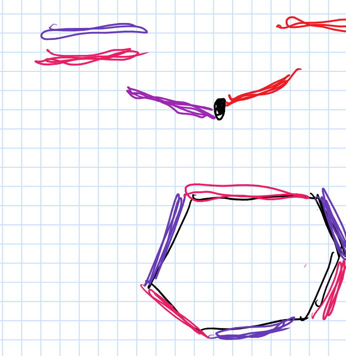


N: Matching
M maximum matching

$N \oplus M = \{ e \in M \cup N \mid e \in M \setminus N \text{ or } e \in N \setminus M \}$



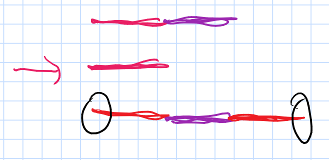
← path alternating



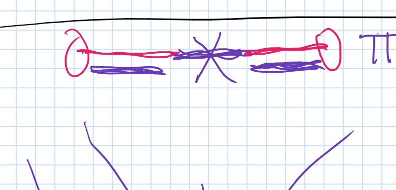
maximum degree 2

even

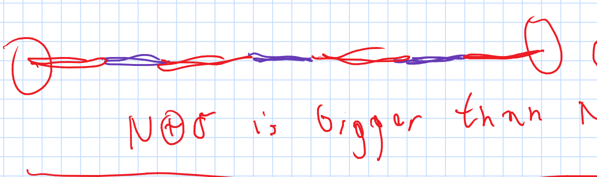
M: maximum matching
N: matching



free (for N)

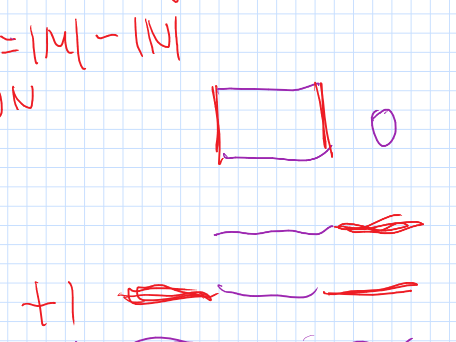


$N \oplus \pi$
augmenting with π



$N \oplus \sigma$ is bigger than N

M: maximum
N: matching
$k = |M| - |N|$
$M \oplus N$



$\pi_1, \pi_2, \ldots, \pi_t$: paths in $M \oplus N$
+1 −1 0
$\delta(\pi) = |\pi \cap M| - |\pi \cap N|$
$\sum_{i=1}^{t} \delta(\pi_i) = k$ $|M| = |N| + k$

⟹ If N is k edges away from being optimal then there are k disjoint augmenting paths

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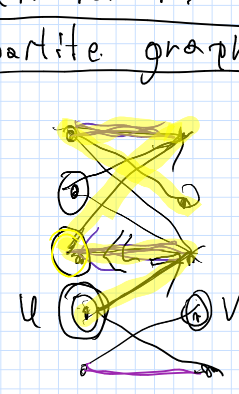
augmenting path for a matching N

π is augmenting if
- start at a free vertex
- ends at a free vertex
- all even edges are in N
  all odd edges are not in N

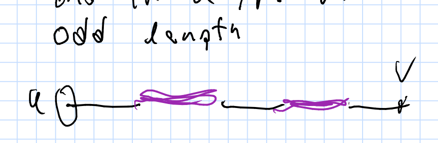
Lemma If N is a matching a π is an augmenting path
⟹ $N \oplus \pi$ is a matching of size $|N| + 1$

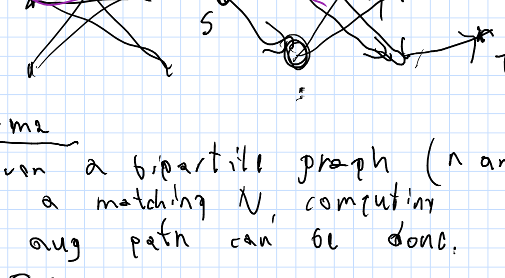
Lemma If N is not a maximum matching then there exists an augmenting path for it.

Bipartite graph | Bipartite matching



augmenting path
alternating
end in a free vertex
odd length





BFS
DFS

Lemma
Given a bipartite graph (n and m) and a matching N, computing an aug path can be done.
$O(n+m)$ time.

$\frac{n}{2}$ augmenting paths

Bipartite matching (G):
  N ← {} // or maximal matching
  $\frac{n}{2}$ [while ∃ aug path for N (say π)
      N ← N ⊕ π] $O(n+m)$
  return N

$O(nm)$: overall running time.

BFS

$O(\sqrt{n})$
$N \oplus$
$N \oplus M \geq \Omega(\sqrt{n})$
$|M| - |N| \leq \sqrt{n}$ n $O(\sqrt{n})$

$O(m\sqrt{n})$ Edmonds and Karp algorithm

$O(n^{2.5})$
$O(n^4)$ 50s