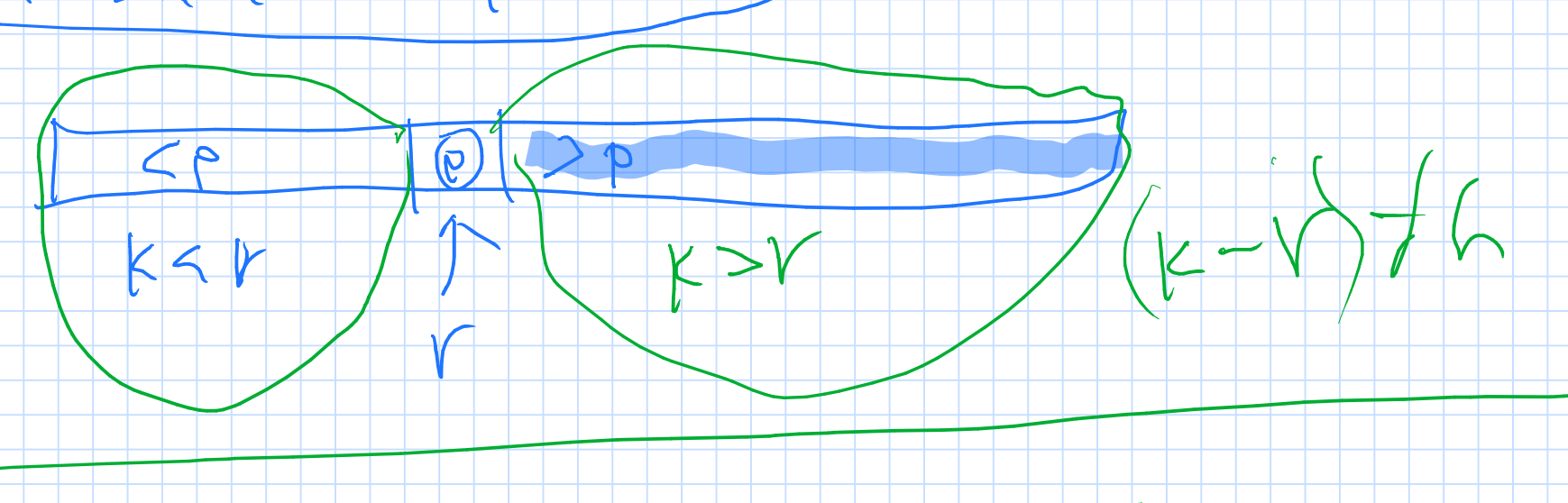
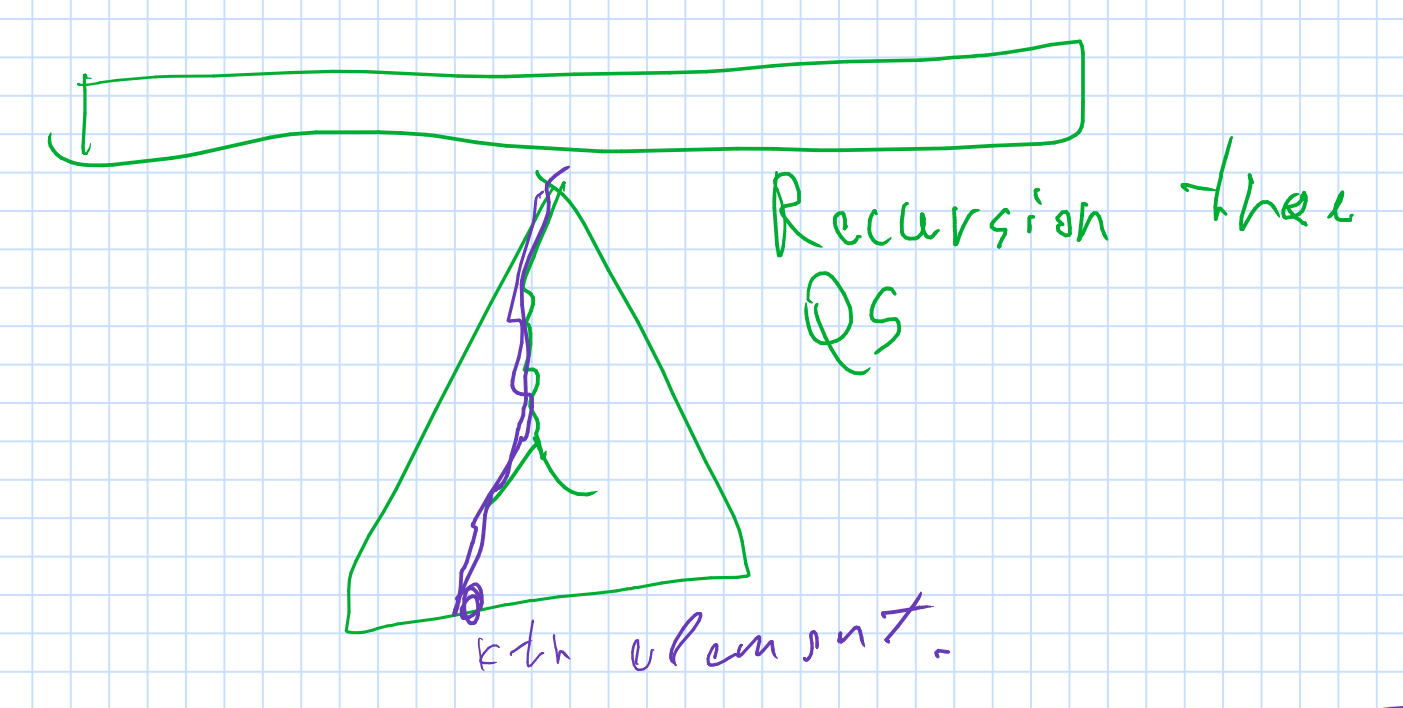
2/11/21 | Lecture 5

QS High probability
Nuts and Bolts
Treaps
Quick Select

Quick Sort
$E[\# \text{comparison}] \leq 2n \ln n$

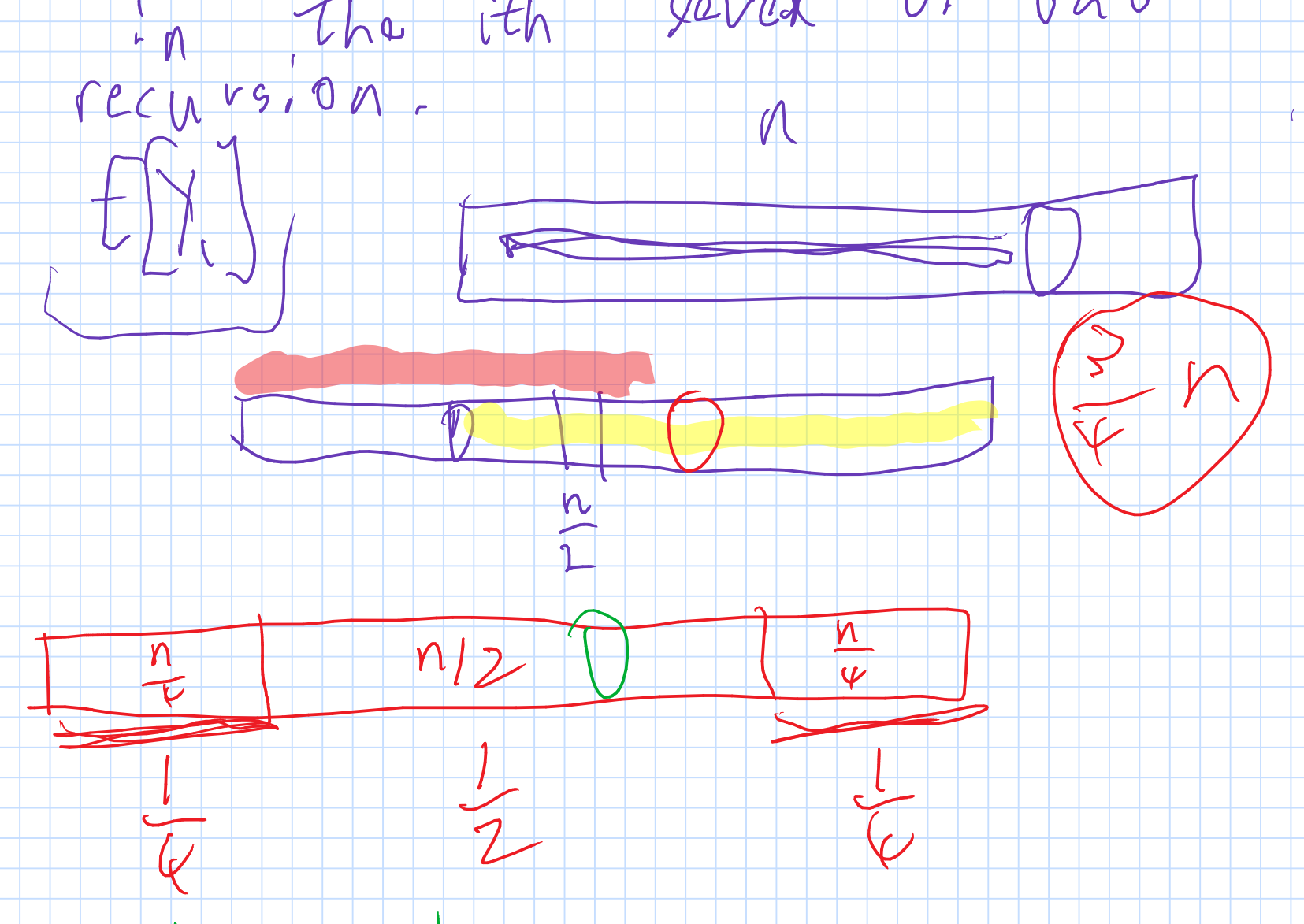
A [ ] n elements
K
output kth smallest number in A
Median selection problem





$Y_i \equiv$ size of the problem in the ith level of the recursion.

$Y_0 = n$ $E[Y_i]$



lucky	screwed
$\frac{1}{2}$	$\frac{1}{2}$
$\leq \frac{3}{4}n$	n

$E[Y_1] \leq \frac{1}{2} \cdot \frac{3}{4} n + \frac{1}{2} n = \frac{7}{8} n$

$E[Y_2] = \left(\frac{7}{8}\right)^2 n$

$E[Y_2 \mid Y_1 = y_1] \leq \frac{7}{8} y_1$

$E[Y_2] = E[E[Y_2 \mid Y_1]] \leq E[\frac{7}{8} Y_1] = \frac{7}{8} E[Y_1] \leq \frac{7}{8} \cdot \frac{7}{8} n = \left(\frac{7}{8}\right)^2 n$

Claim
$E[Y_i] \leq \left(\frac{7}{8}\right)^i$. for all i.
Proof: Induction. QED

Lemma
The expected running time of Quick Select is O(n).
Proof
The running time
$O(Y_1 + Y_2 + Y_3 + \cdots +)$
$E[RT] = O(E[\sum Y_i])$
$E[\sum Y_i] = \sum_{i=0}^{\infty} E[Y_i] \leq \sum_{i=0}^{\infty} \left(\frac{7}{8}\right)^i n \leq O(n)$. ∎

Claim
QuickSelect performs at most 20 log n recursive calls before terminating. with probability $\geq 1 - 1/n^2$.

$\ln x = -\ln \frac{1}{x}$

Proof
Y_h

$h = \lceil 4 \log_{\frac{8}{7}} n \rceil + 1$

$\log_{\frac{8}{7}} n$

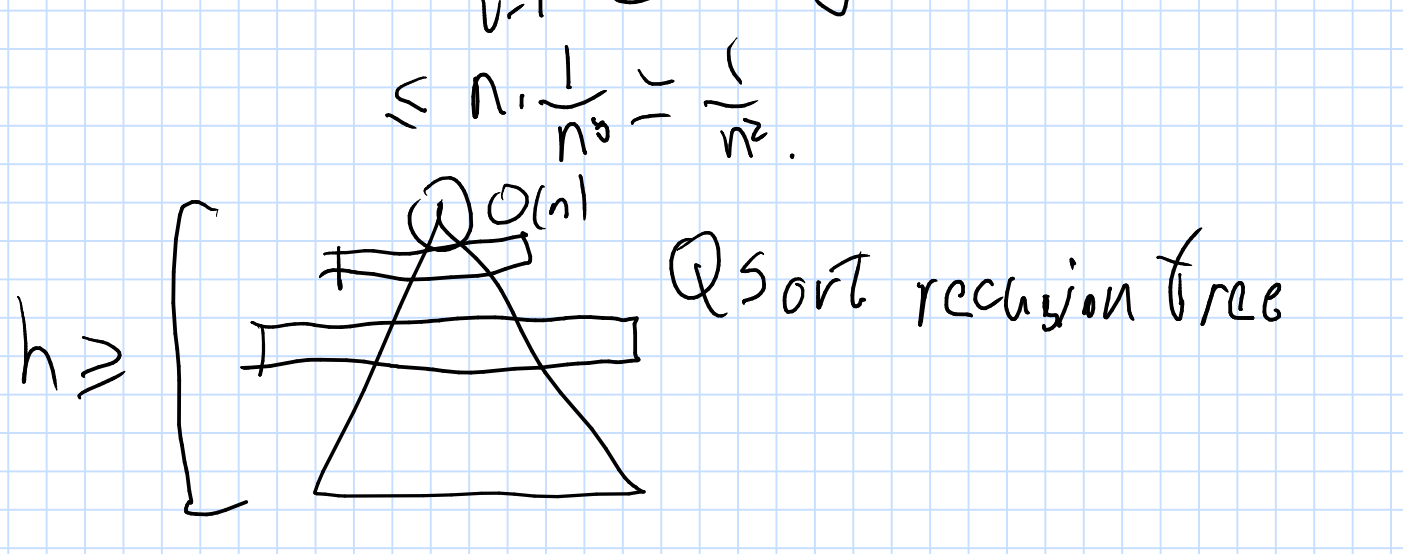
$u = \log_{\frac{8}{7}} n$ $E[Y_u] \leq \left(\frac{7}{8}\right)^u n = \exp\left(\ln \frac{7}{8} \frac{\ln n}{\ln 8/7}\right) n$
$= \exp\left(\ln \frac{7}{8} \frac{\ln n}{-\ln \frac{7}{8}}\right) = e^{-\ln n} = \frac{1}{n}$

$h = 4u$ $E[Y_h] = \left(\frac{7}{8}\right)^h n \leq \left(\frac{7}{8}\right)^{4u} n$
$= \left(\frac{1}{n}\right)^4 \cdot n = \frac{1}{n^3}$.

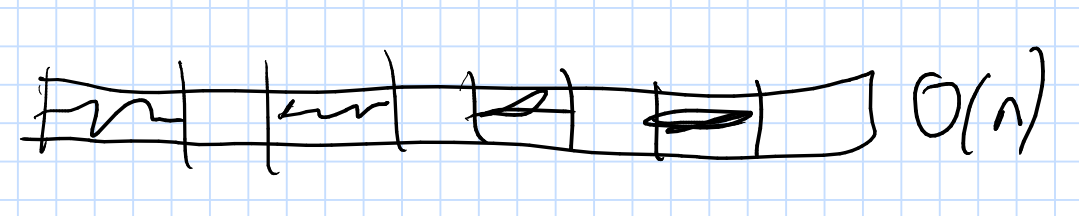
P_r[QuickSelect recursion depth exceeds h]
$\leq P_r[Y_h \geq 1] \leq \frac{E[Y_h]}{1} \leq \frac{1}{n^3}$. ∎

Lemma QuickSort performs at most $h \cdot n = (4 \log_{\frac{8}{7}} n) n$ comparisons with probability $\leq \frac{1}{n^2}$.
Proof
X_i = number of recursive calls that the ith smallest number in A participates in during the execution of Quick Sort.

$i = 1, \ldots, n$
$P[X_i \geq h] \leq \frac{1}{n^3}$
$P[\forall i: X_i < h] = 1 - P[\exists i: X_i > h] \geq 1 - \frac{1}{n^2}$.
$P[\exists i: X_i \geq h] \leq P[\bigcup_i (X_i \geq h)]$ union bound
$\leq \sum_{i=1}^{n} P[X_i \geq h]$
$\leq n \cdot \frac{1}{n^3} = \frac{1}{n^2}$.

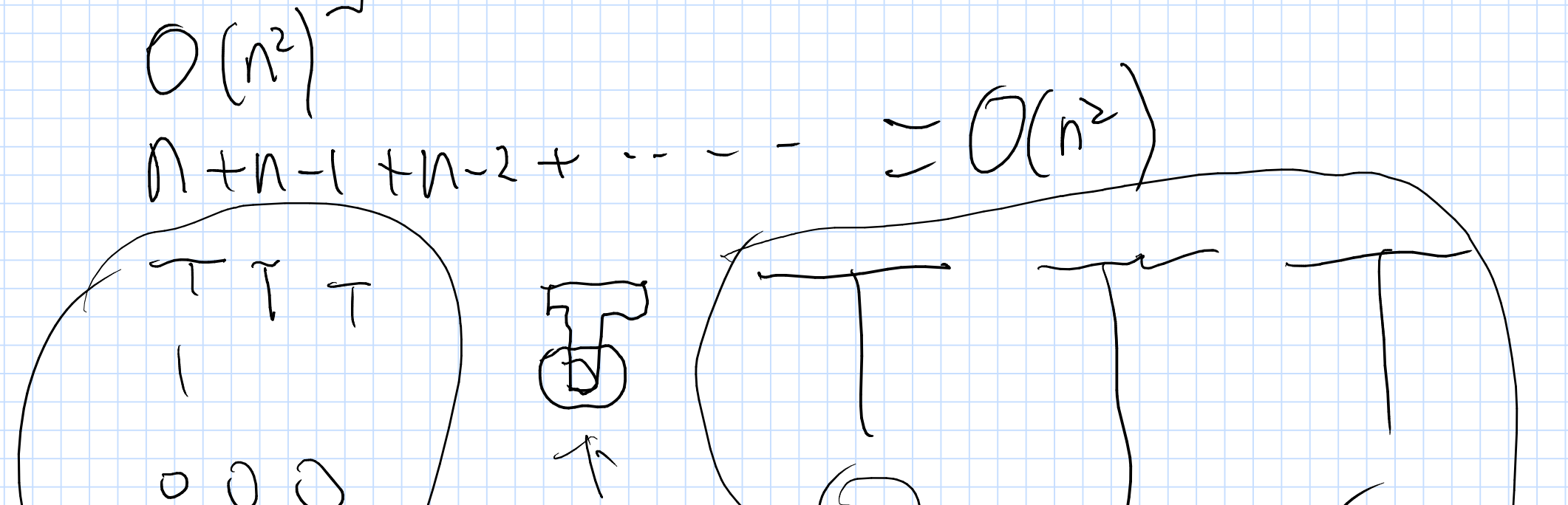


$n_1 + n_2 + 1 = n$
$O(n_1) + O(n_2) = O(n)$



$O(hn)$ running time.

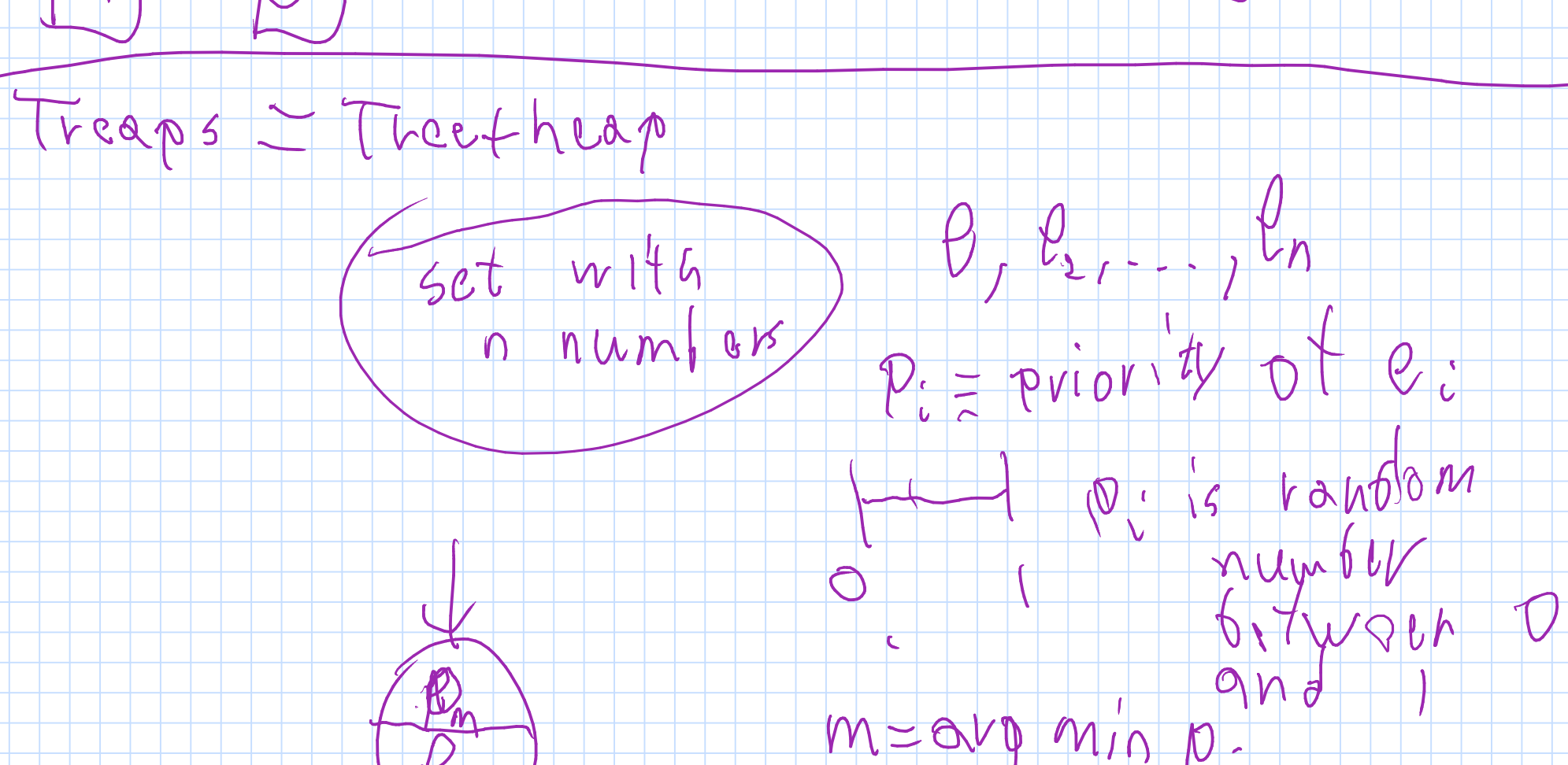
Nuts and bolts
n nuts, n bolts. n perfect pairs
$O(n^2)$
$n + n-1 + n-2 + \cdots = O(n^2)$



Quick sort for nuts and bolts.

O(n log n)

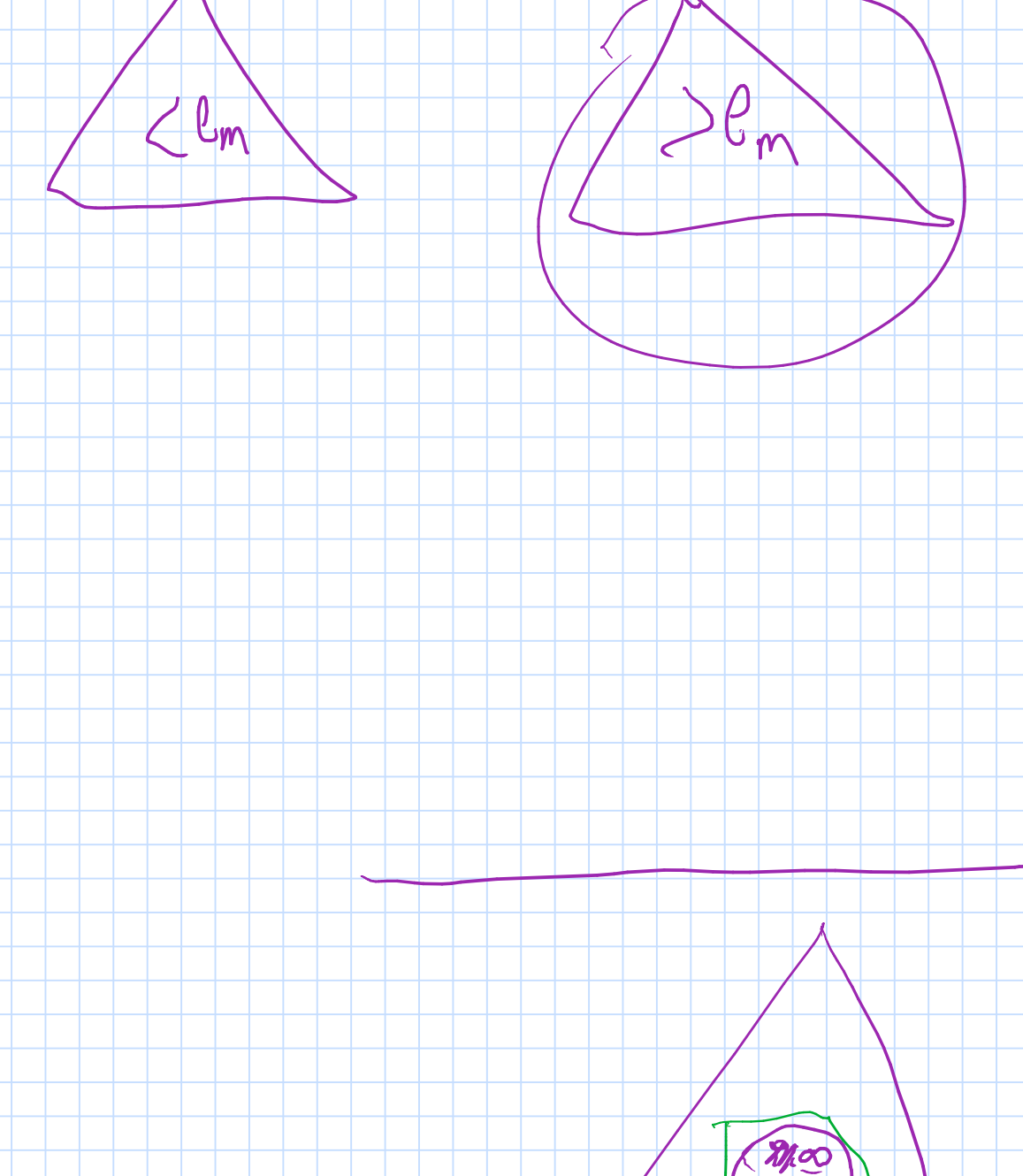
Treaps

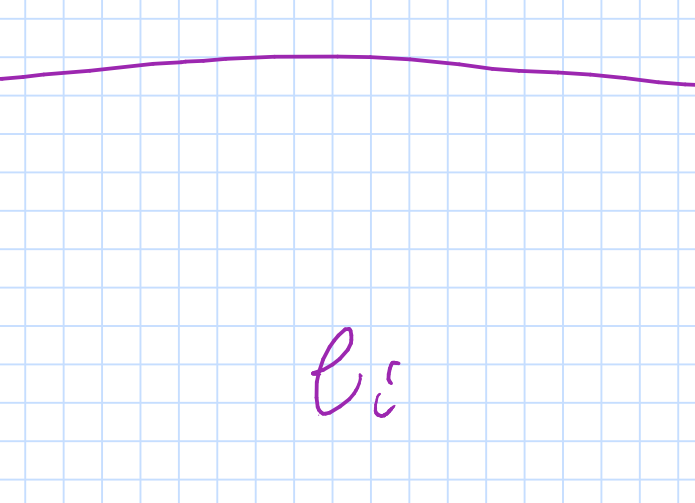


Treaps = Tree+heap

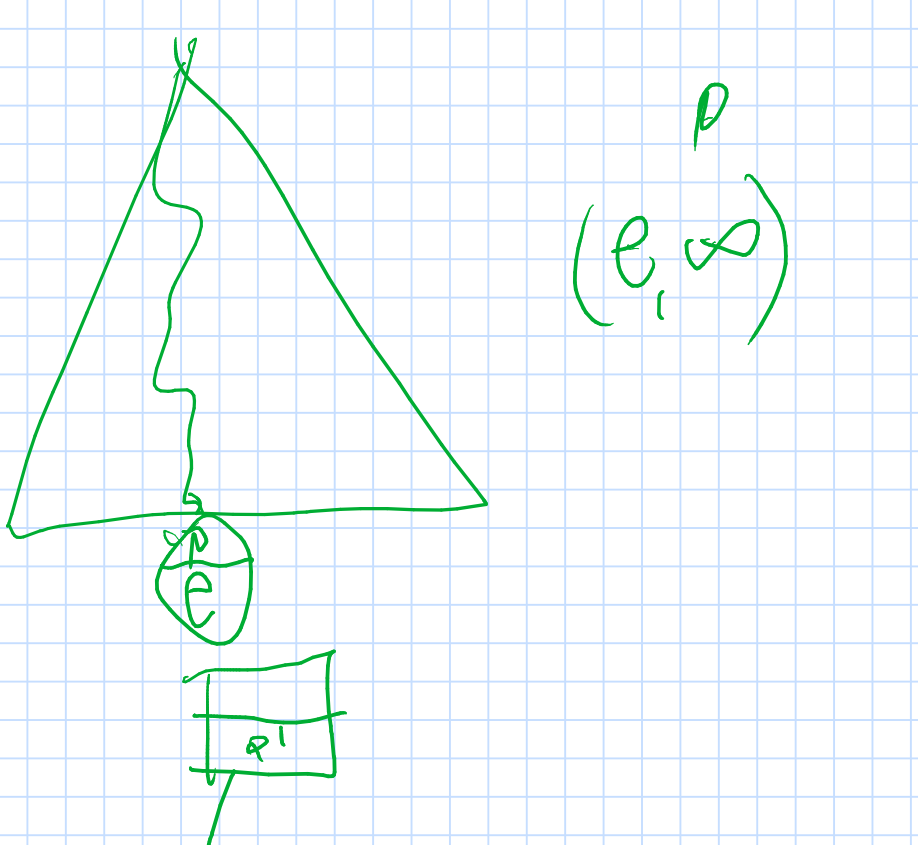
set with n numbers

$e_1, e_2, \ldots, e_n$
p_i = priority of e_i
p_i is random number between 0 and 1
$m = \arg\min_i p_i$



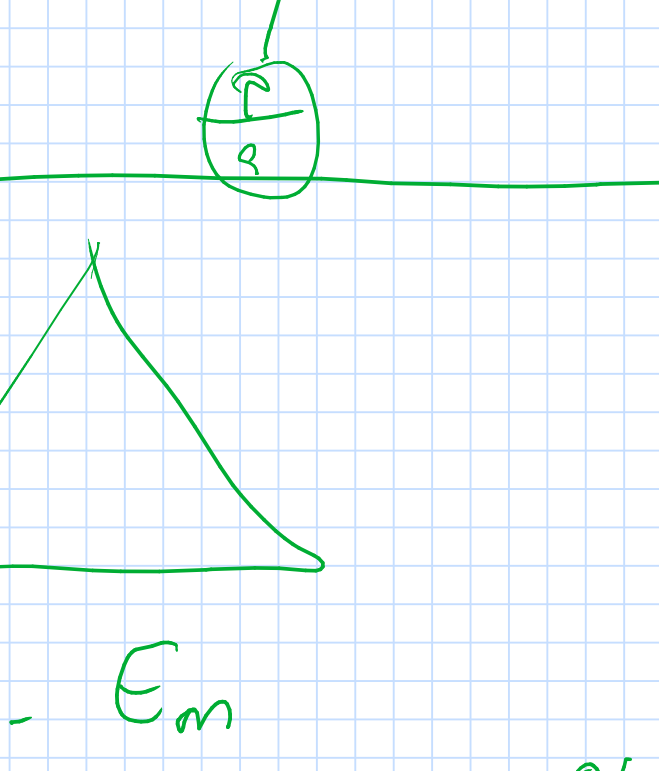


Same depth as the recursion tree of Quick Sort
$\geq 1 - \frac{1}{n^2}$
$h \leq O(\log n)$

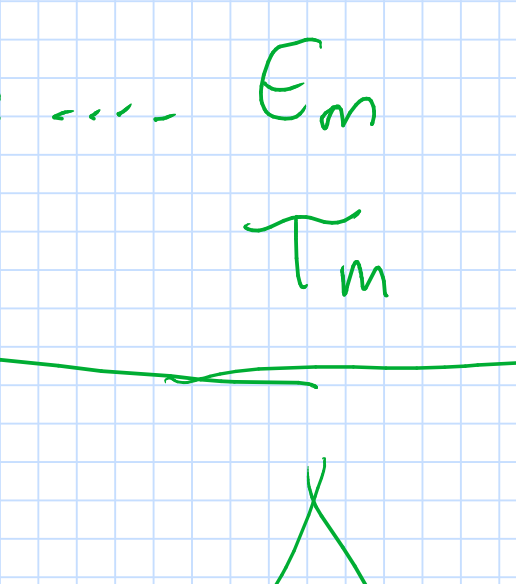


e_i

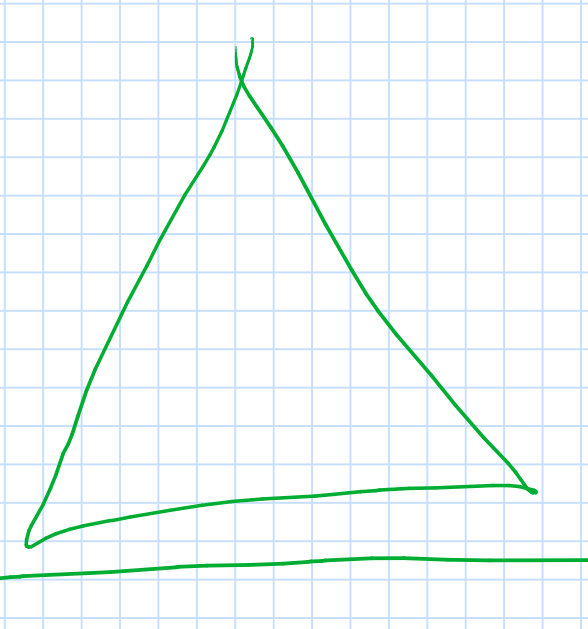
Insertion



(e, ∞) p



$E_1, E_2, E_3, \ldots, E_m$
T_m $m = O(n^2)$



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