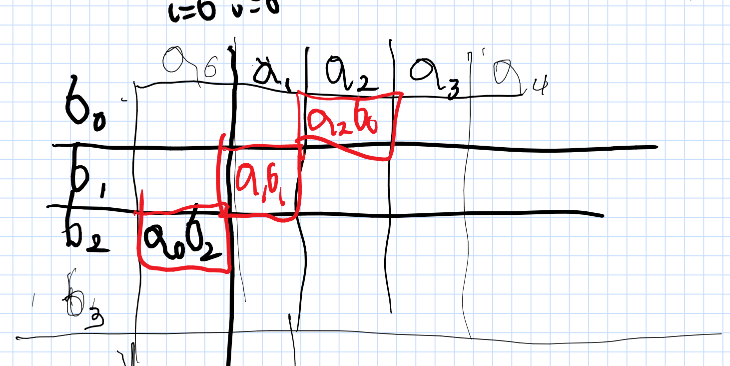
1/26/21

FFT: Fast Fourier Transform

$p(x) = \sum_{i=0}^{n-1} a_i x_i$ $\quad a_0, a_1, \ldots, a_{n-1}$

$q(x) = \sum_{i=0}^{n-1} b_i x^i$ $\quad b_0, b_1, \ldots, b_{n-1}$

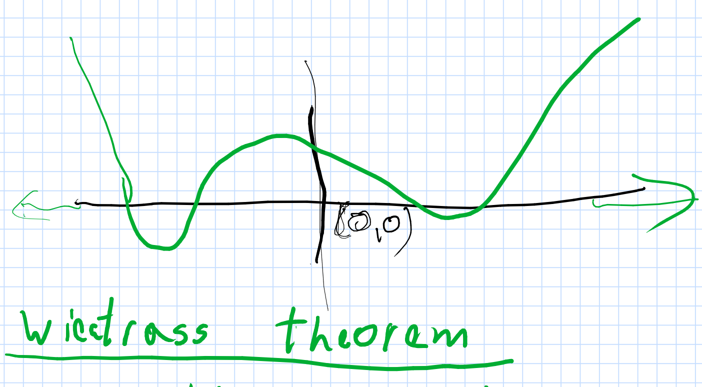
$p(x)\cdot q(x) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} a_i b_j x^{i+j}$ $\quad O(n^2)$



$\ldots \quad (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 \ldots \equiv pq$

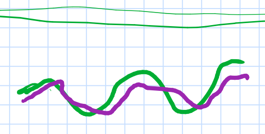
Kartsuba alg $O(n^{\log_2 3})$

$O(n \log n)$ time FFT



Wiectrass theorem

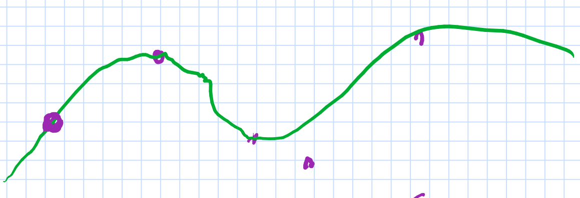
Any continous function on a close interval can be approx by a polynomial



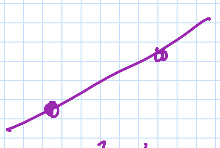
Taylor exp

$f(x) = \sum \alpha_i x^i$ ←

$(x_0, y_0), (x_1, y_1), \ldots (x_{n-1}, y_{n-1})$



$a_0 + a_1 x$



point value representation

$(x_0, y_0) \ldots (x_i, y_i) \ldots$

$x = x_i \quad (x-x_1)(x-x_2)\cdots(x-x_{i-1})$ [scribbled out] $(x-x_{i+1})\cdots(x-x_{n-1}) \equiv P_i$

$x = x_i$

$P_i(x) = 0 \quad \forall x \in \{x_0, x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_n\}$

$P_i(x_i) = \prod_{j \neq i} (x_i - x_j) \neq$

$$g_i \equiv y_i \frac{P_i(x)}{P_i(x_i)} = \begin{cases} y_i & x = x_i \\ 0 & x \in \{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots x_{n-1}\} \end{cases}$$

$(x_0, y_0), (x_1, y_1) \ldots (x_{n-1}, y_{n-1})$

$g(x) = \sum_{i=0}^{n-1} g_i(x) \quad n-1 \quad O(n^2)$

$\sum \alpha_i x^i$

$g(x) \quad x_0, x_1, \ldots, x_{n-1}$

$O(n^2) \quad (x_0, g(x_0)), \ldots (x_i, g(x_i))$

$p \equiv \{(x_0, y_0) \ldots (x_m, y_m)\}$

$q \equiv \{(x_0, z_0), \ldots, (x_m, z_m)\}$

$pq \equiv \{(x_0, p(x_0)\cdot q(x_0)), \ldots$ $\quad O(m)$

(x_0, y_0, z_0) $\}$

Coefficient rep ⇒ point-value pair rep

$O(n^2)$

p, q degree n ⇒ degree $2n$

$m = 2n$ ←

$2n$

$n = 2^i$

Idea divide and conquer

Coefficients rep ⇒ point-val pair rep

$p(x) = \sum a_i x_i$

$[a_0, a_1, \ldots, a_{n-1}] \quad [x_0, x_1, \ldots, x_{n-1}]$

$(x_0, p(x_0)) \ldots (x_i, p(x_i))$

$n \quad \frac{n}{2}$

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots a_n x_n$

$P_1(x) = a_0 + a_2 x^2 + \ldots + a_4 x^4 \quad - x^2$

$P_2(x) = a_1 x + a_3 x^3 + \ldots$

$\hat{P}_1(y) = \sum a_{2i} y^i \quad P_1(x) = \hat{P}_1(x^2)$

$\hat{P}_2(y) = \sum_{i=0}^{n/2} a_{2i+1} y^i \quad P_2(x) = \hat{P}_2(x^2) x$

$P(x) = P_1(x^2) + x P_2(x^2) \quad x_0, x_1, \ldots, x_{n-1}$

$[P_1(\langle x_0^2, x_1^2, x_2^2 \ldots x_{n-1}^2)]$

$[P_2(\langle x_1^2 \ldots x_{n-1}^2)]$

For $i=0$ to $n-1$ do
$\quad y_i \leftarrow P_1(x_i^2) + P_2(x_i^2) x_i$ $\quad O(n)$

$x_0, x_1, x_2, \ldots, x_{n-1}$

$x_0, x_2, x_4, \ldots x_{2i} \ldots > 0$ some positive real numbers

$x_{2i+1} = -x_{2i}$

$1, -1, 2, -2, \ldots$

$P_1\ P_2 \quad \overbrace{\{x_0^2, x_1^2, x_2^2, \ldots x_{n-1}^2\}}^{n/2 \text{ values}} =$

degree $n/2$

A set X of values collapse if $X^2 = \{x^2 \mid x \in X\}$ $\quad |X^2| = |X|/2$

Assume we have a set that keeps collapsing

$X^{2^i} = \{x^{2^i} \mid x \in X\}$

$|X^{2^i}| = \frac{n}{2^i} \quad n$ power of two

$\text{FFT}(p, X = \{x_0, x_1, \ldots x_{n-1}\})$
$\quad P_1, P_2 \leftarrow$ split p
$\quad y = X^2$ ←
$\quad \text{FFT}(P_1, y) \quad n/2 \quad n/2$
$\quad \text{FFT}(P_2, y) \quad n/2 \quad n/2$
$\quad$ For $i=0$ to $n-1$
$\quad\quad p(x_i) = P_1(x_i^2) + P_2(x_i^2) \cdot x_i$ $\quad O(n)$
$\quad$ return $(x_0, p(x_0)), \ldots (x_{n-1}, p(x_{n-1}))$.

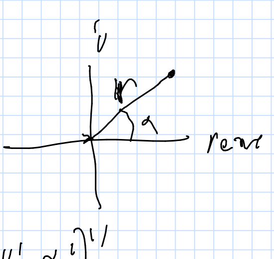
$T(n) = O(n) + O(n) + 2T(\frac{n}{2}) + O(n)$

$T(n) = 2T(\frac{n}{2}) + O(n) = O(n \log n)$

Big trick complex numbers

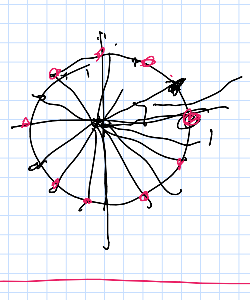
$a + ib$

$i^2 = -1$



$"(r, \alpha)" \cdot "(r', \alpha')"$
$= (rr', \alpha + \alpha')$

$r = 1$



$\frac{2\pi}{n}$ $\quad n =$ power of two

n roots of unity