Assignment 2 (due Feb 23 Friday 2pm (in class))

You may work in a group of at most 3 students. Carefully read [https://courses.engr.illinois.edu/cs498tc3/policies.html](https://courses.engr.illinois.edu/cs498tc3/policies.html) and [https://courses.engr.illinois.edu/cs498tc3/integrity.html](https://courses.engr.illinois.edu/cs498tc3/integrity.html).

1. **[30 pts]** A box (or a hyper-rectangle) is the higher-dimensional generalization of a rectangle. In 3D, a box has 8 vertices, 12 edges, and 6 faces, with edges parallel to the \(x\)-, \(y\)-, and \(z\)-axes. Let \(S\) be a set of \(n\) boxes in 3D, where each box has the origin as a vertex, i.e., each box is of the form \([0, a_i] \times [0, b_i] \times [0, c_i]\). The union \(U\) of the \(n\) boxes in \(S\) is a nonconvex polyhedron. We are interested in computing this polyhedron \(U\). (This problem, in 3 dimensions and higher, has many applications.)

   (a) **[5 pts]** In 3D, show that \(U\) has \(O(n)\) vertices, edges, and faces. What does a face look like?

   (b) **[7 pts]** Consider an incremental approach to computing \(U\). Prove that if we randomize the order of insertion of the boxes, the expected total number of vertices, edges, and faces created and destroyed is \(O(n)\).

   (c) **[13 pts]** Again consider an incremental approach. This time, describe a simple insertion order, without randomization, for which the total number of vertices, edges, and faces created and destroyed is guaranteed to be \(O(n)\). Using this insertion order (and appropriate data structures), describe a deterministic \(O(n \log n)\)-time algorithm to compute the polyhedron \(U\) in 3D.

     (Note: alternatively, such an algorithm can be obtained by a sweep approach. What direction should the sweep be in?)

   (d) **[5 pts]** If we do not assume that each box has the origin as a vertex, does \(U\) still have \(O(n)\) vertices, edges, and faces? Explain.

2. **[15 pts]** Given \(n\) circular disks of the same radius, give an \(O(n \log n)\)-time algorithm to compute the area of the union of these disks. (Note: the union may be disconnected and may have holes, but we do not need to output the union explicitly, just its area. Hint: what does the union look like inside a Voronoi cell?)