Announcements

• Work in groups - groups of 2 for MPs, groups of 2-4 for the final project.
• MP1 is graded
• MP2 is out, MP2.1 is due on next Monday, Sep12@11am
  MP2.2 due Monday, Sep19@11am
  MP2.3&4 due Monday, Sep26@11am
• Final projects ideas: in class and more on Piazza@post83
  https://piazza.com/class/is0v9gwpql6517?cid=83
• Currently, project proposals are due on Sep 28.
Announcements

Project team and abstract due Sep 28, 11:59pm.

Two types of projects
  • supervised
  • unsupervised

Do not wait until the last minute (day, week) - professors are busy! Start contacting professors today.
VR Paddle Boarding

Contact: mhermand@illinois.edu
prof. Manuel Hernandez, Assistant Professor, Kinesiology & Community Health, Neuroscience Program.
Virtual Classroom Experiment

Would you learn better from a female or male professor? How does gender bias affect students learning in the classroom?

Contact: Anna Yershova
Unit Quaternions: Compositions, Inverses and Duplicates

\[ q = \left( \cos \frac{\theta}{2}, \ v_1 \cdot \sin \frac{\theta}{2}, \ v_2 \cdot \sin \frac{\theta}{2}, \ v_3 \cdot \sin \frac{\theta}{2} \right) \]
Unit Quaternions: Compositions, Inverses and Duplicates.

\[ q = \left( \cos \frac{\theta}{2}, \ v_1 \cdot \sin \frac{\theta}{2}, \ v_2 \cdot \sin \frac{\theta}{2}, \ v_3 \cdot \sin \frac{\theta}{2} \right) \]

\[ (\theta, \vec{v}) \]

\[ (-\vec{v}, \ ) \]

\[ (\vec{v}, \ ) \]

\[ q = \]
Representation of Rotations: Unit Quaternions

\[ q = (a, b, c, d) \in \mathbb{R}^4, \quad a^2 + b^2 + c^2 + d^2 = 1 \]

The set of all unit \( q \) is a hypersphere \( (S^3) \)

\( S^2 \) lives in \( \mathbb{R}^3 \)
\( S^1 \) lives in \( \mathbb{R}^2 \)
\( S^0 \) lives in \( \mathbb{R}^1 \)

In Unity 3D:
\[ (x, y, z, \omega) = (b, c, d, a) \]

In math:
\[ a + bi + cj + dk \]
Unit Quaternions: Compositions, Inverses and Duplicates

\[ q = \left( \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \cos \frac{\vec{v}}{2}, \sin \frac{\theta}{2} \sin \frac{\vec{v}}{2} \right) \]

\[ \iff \]

\[ (\theta, \vec{v}) \]

\[ q^{-1} = \]
Unit Quaternions: Compositions, Inverses and Duplicates

\[ q = \left( \frac{a}{\cos \Theta}, \frac{b}{\sin \frac{\Theta}{2}}, \frac{c}{\sin \frac{\Theta}{2}}, \frac{d}{\sin \frac{\Theta}{2}} \right) \]

What is the inverse of \( q \)?

\[ q^{-1} = (-a, -b, -c, -d) \]

What is the duplicate of \( q \)?

\[ q^\perp = (-a, b, c, d) \]

\[ q^\perp = (a, -b, -c, d) \]
Unit Quaternions: Compositions, Inverses and Duplicates

\[ q = \left( \cos \frac{\Theta}{2}, \ v_1 \cdot \sin \frac{\Theta}{2}, \ v_2 \cdot \sin \frac{\Theta}{2}, \ v_3 \cdot \sin \frac{\Theta}{2} \right) \]

\[ q_1 = q_1^{-1} \cdot q_{23} \cdot q_{12} \cdot q_1 \]

\[ q_3 = q_{23} \cdot q_{12} \cdot q_1 \]
Unit Quaternions: Multiplication

\[ q_1 = (a_1, b_1, c_1, d_1) \]
\[ q_2 = (a_2, b_2, c_2, d_2) \]
\[ \overrightarrow{p}_1 = (b_1, c_1, d_1) \]
\[ \overrightarrow{p}_2 = (b_2, c_2, d_2) \]

\[ q_1 \circ q_2 = (a_1 a_2 - \overrightarrow{p}_1 \cdot \overrightarrow{p}_2, \overrightarrow{p}_1 \times \overrightarrow{p}_2 + a_1 \overrightarrow{p}_2 + a_2 \overrightarrow{p}_1) \]

+ renormalize

Order of operations: \( q_1 \circ q_2 \neq q_2 \circ q_1 \).

Inverses: \( (q_1 \circ q_3 \circ q_2 \circ q_1)^1 = \)

Efficiency: Haar Measure, only 4 parameters
Steve is a Minecraft character. His head is a cube. The center of his head is the origin of the GLOBAL coordinate frame, in which his left pupil has coordinates (1, 0, 3).

Calculate the coordinates of Steve's left pupil after Steve's head is turned first by a yaw of 90 degrees followed by a roll by 90 degrees in GLOBAL coordinate frame.
Chaining Matrices in Global Coordinate Frame

1. \( \mathbf{p} = (1, 0, 3) \), \( \mathbf{p}' = (\quad) \), \( \mathbf{p}'' = (\quad) \)

Pupil:

\[
\begin{align*}
\mathbf{R}_y(\frac{\pi}{2}) &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \\
\mathbf{R}_z(\frac{\pi}{2}) &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\end{align*}
\]
Applying Quaternion Rotation to a Vector

Vector \((x, y, z) \in \mathbb{R}^3\)

Rotate by quaternion \(q\)

\[ p = (x, y, z, 1) \]

\[ p' = q \circ p \circ q^{-1} \]
Characterizing Object Motion

"Natural" with respect to local coordinate frame
Steve is a Minecraft character. His head is a cube. Originally, his LOCAL coordinate frame coincides with the GLOBAL coordinate frame and his left pupil has coordinates (1, 0, 3).

Calculate the coordinates of Steve's left pupil after Steve turns his head first by a yaw of 90 degrees and then by a

Answer:
Chaining Matrices in Local Coordinate Frame

1. $R_y\left(\frac{\pi}{z}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

2. $R(\text{?}) = \begin{bmatrix} \text{?} & \text{?} \\ \text{?} & \text{?} \end{bmatrix}$

\[
\begin{bmatrix}
[1 \\
0]
\end{bmatrix}
\begin{bmatrix}
[1 \\
0 \\
3]
\end{bmatrix} = 
\begin{bmatrix}
0 \\
3 \\
-1
\end{bmatrix}
\]
Matrix Multiplication Property: **Associativity**

\[A \cdot B \cdot C \cdot D = A \cdot (B \cdot C) \cdot D\]

\[R_1 \cdot R_2 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_1 \cdot (R_2 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix})\]
Transformations: Where are we?

Matrices

2x2
- Rotations
- Shear
- Scale
- Projection
- Reflection

3x3
- Rotations
- Yaw
- Pitch
- Roll
- Shear
- Projection
- Scale
- Reflection

4x4

3D Rotations
- Axis-Angle
- Exponential coordinates
- Quaternions

Translations using vectors
Limitations of 3x3 Matrices

\[
\begin{bmatrix}
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
= 
\begin{bmatrix}
x' \\
y' \\
z' \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
= 
\begin{bmatrix}
\end{bmatrix}
\]
Chaining Translations and Rotations

Rotate by $R$, then translate by $t = (t_x, t_y, t_z)$

$$\begin{bmatrix} R \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Place parentheses in proper places

Translate by $t = (t_x, t_y, t_z)$, then rotate by $R$

$$\begin{bmatrix} R \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$
Homogeneous Transformations: DOFs?

Rotate by $R$, then translate by $\vec{t} = (t_x, t_y, t_z)$.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = R\vec{x} + \vec{t}$$

Describes ALL possible rotations and translations of a 3D rigid body (3D rigid transformations).

Sanity check:

for $\vec{t}$: total 9

for $R$: total 3
Homogeneous Transformation Matrix

Rotate by $R$, then translate by $t = (t_x, t_y, t_z)$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Have: $x' = Rx + t$

Want: $x' = Ax$

Solution: Algebraic trick

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$