

CS 498mp3: Logic in Computer Science Spring 2017: Homework 1
Thursday 9:30am, February 16th
Hand over in class, before lecture begins.

1. **Proof by structural induction** [20 points]

Consider a formal syntax for well-formed propositional logic give as:

$$\text{Formulas } \alpha, \beta ::= p_i \mid (\neg\alpha) \mid (\alpha \vee \beta) \mid (\alpha \wedge \beta)$$

where $P = \{p_1, p_2, \dots\}$ is an infinite set of propositions, and where $i \in \mathbb{N}$ above.

Prove formally that in any formula α , if c is the number of binary propositional connectives (\vee and \wedge) and n is the number of occurrences of propositions in α , then $n = c + 1$. Prove this by structural induction on formulas.

2. **Resolution** [20 points]

Prove the following is valid using resolution:

$$((p \Rightarrow q) \wedge (r \Rightarrow s)) \Rightarrow ((p \vee r) \Rightarrow (q \vee s))$$

3. **Understanding König's Lemma** [30 points]

There is a gigantic box with balls, each marked with a natural number. There are infinitely many balls in this box, and in fact infinitely many balls marked with the same number.

A robot (called Golem, if you must know, and yes, he is made of clay) sets out by pulling out a single ball with some number n , and places it in his basket. Now, in every round, he takes one ball from his basket, say with a number i on it, and puts it back in the box, and takes out *any* number of balls from the box with the condition that all of them have the same number on them, but this number is less than i .

For example, in a round, he may transfer a ball numbered 5 from the basket to the box, and take back 30 balls labeled 4. Or put one ball numbered 5 into the box and take back 3413343 balls labeled 3. His basket has unbounded capacity.

Note that if Golem transfers a ball numbered 0 from the basket to the box, he cannot take back any ball.

Prove formally that no matter how Golem goes about picking the balls, he will eventually empty his basket.

Hint: Use König's Lemma.

4. Modeling using propositional logic [30 points]

- (a) Three boxes are in a room; let's call them Ouro, Zoloto, and Thangam. One contains gold, the other two are empty. Each box has imprinted on it a clue as to its contents; the clues in each box are:

(Ouro) “The gold is not in this box”,

(Zoloto) “The gold is not in this box”,

(Thangam) “The gold is in Box Zoloto”.

Only one message is true; the other two are false.

You want to find out which box contains the gold.

Model the problem as a satisfiability problem in propositional logic. Use three variables O , Z , T , to denote whether the gold is in the Ouro, Zoloto, or Thangam box, respectively. Use three other variables to denote the truthhood of each of the labels on the boxes. Model all constraints using formulas, and feed them to a SAT solver to find what's true.

Present both your hand-written constraint and the SAT solver's answer.

Note: You can use Z3 for SAT solving online at <http://www.rise4fun.com/z3>. Here's a sample syntax for checking whether $(p \vee q) \wedge (p \Rightarrow (\neg q)) \wedge (q \Rightarrow (\neg p))$ is satisfiable, and to ask Z3 to give you a model.

```
(declare-const p Bool)
(declare-const q Bool)
(assert
  (and
    (or p q)
    (=> p (not q))
    (=> q (not p))
  )
)
(check-sat)
(get-model)
```

- (b) Using a similar technique as above, model the following in propositional logic, using an appropriate set of propositions, and solve it using Z3.

There are three suspects for a murder: Adams, Brown, and Clark. Adams says “I didnt do it. The victim was an old acquaintance of Brown's. But Clark hated him.” Brown states “I didn't do it. I didn't even know the guy. Besides I was out of town all that week.” Clark says “I didn't do it. I saw both Adams and Brown downtown with the victim that day; one of them must have done it.” Assume that the two innocent men are telling the truth, but that the guilty man might not be. Who did it?