Tuesday, May 02, 2017
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CS 498mp3: Logic in Computer Scinece, Spring 2017, MIDTERM EXAM

LASTNAME, FIRSTNAME (in CAP letters): **NETID:**

Problem	1	2	3	4	5	6	7	Total
Possible	20	16	10	14	18	10	12	100
Score								

- Exam has 8 sheets (16 pages); please check you have a complete exam.
- Time for exam: 1 hour 15min
- Write on both sides of the sheets.
- No notes/textbook/etc. allowed.

1. **[20 points (varying points for questions)]**

(a) The set of all valid FOL formulas over any finite signature is

- Decidable ☐
- Recursively enumerable and undecidable ☑
- Not recursively enumerable and undecidable ☐

(b) For any FO model M over a signature, $Th(M)$ is decidable.

True ☐ False ☑

(c) There is a model M such that $Th(M)$ is recursively axiomatizable.

True ☑ False ☐

(d) Let T be a complete theory. Then T is .

- Decidable ☐
- Recursively enumerable and undecidable ☐
- Not recursively enumerable and undecidable ☑

(e) Let T be a theory and let $T \models \varphi$ for some sentence φ. Then

- φ is true in all models of T ☑
- φ is true in some model of T ☐
- $T \vdash \varphi$ system (in a sound and complete proof system) ☑

(f) Let T be the theory of natural numbers with addition and multiplication. Then T is recursively axiomatizable.

True ☐ False ✓

(g) Let M be a model. Then $Th(M)$ is complete.

True ✓ False ☐

(h) Let T be a consistent set of sentences and let $T \models \varphi$ for a sentence φ. Then

$T \cup \{\varphi\}$ is necessarily consistent ✓

$T \cup \{\neg\varphi\}$ is necessarily consistent ☐

$T \cup \{\varphi\}$ is necessarily inconsistent ☐

$T \cup \{\neg\varphi\}$ is necessarily inconsistent ✓

$\varphi \in T$ is necessarily true ☐

(i) Let T be a set of sentences that is decidable. Then T has a recursive axiomatization.

True ✓ False ☐

(j) Let T be a set of sentences and let $T \models \varphi$ for a sentence φ. Then which of the following are necessarily true?

There is a finite subset T' of T such that $T' \models \varphi$ ✓

There is a finite subset T' of T such that $T' \cup \{\varphi\}$ is inconsistent. ☐

There is a finite subset T' of T such that $\models \left(\bigwedge_{\psi \in T'} \psi\right) \wedge \varphi$. ☐

There is a finite subset T' of T such that $\models \left(\bigwedge_{\psi \in T'} \psi\right) \Rightarrow \varphi$. ✓

2. **[4+4+4+4=16 points]: Short questions**

(a) **Expressing properties in FOL** [5 points]

A coloring of a finite or infinite graph using the three colors *Red*, *Green*, and *Blue* is a mapping that maps each node to a color such that adjacent nodes do not have the same color.

Over the signature that has a single binary edge-relation E and three unary relations *Red*, *Green*, and *Blue*, and assuming that models satisfy the axiom $\forall x, y.E(x,y) \Leftrightarrow E(y,x)$, write down a sentence φ such that φ is true precisely in those models where the three unary relations define a coloring of the graph defined by E.

You do not have to prove your formulation is correct. Just give the sentence.

$\forall x\ (Red(x) \vee Green(x) \vee Blue(x)$
$\wedge\ Red(x) \Rightarrow (\neg Green(x) \wedge \neg Blue(x))$
$\wedge\ Green(x) \Rightarrow (\neg Red(x) \wedge \neg Blue(x))$
$\wedge\ Blue(x) \Rightarrow (\neg Red(x) \wedge \neg Green(x)))$
$\wedge\ \forall x,y\ (E(x,y) \Rightarrow$
$(\neg (Blue(x) \wedge Blue(y))$
$\wedge \neg (Red(x) \wedge Red(y)) \wedge \neg (Green(x) \wedge Green(y))))$

(b) Give a model whose theory is undecidable.

$(\mathbb{N}, +, *, 0, 1)$

where $+, *, 0, 1$ are the standard definitions.

(c) Write a FO sentence that is true over $(\mathbb{R}, +, \times, \leq)$ but not true over $(\mathbb{R}^+, +, \times, \leq)$. The first is the universe of real numbers and the second is the universe of non-negative real numbers, and the functions and relations are interpreted in the usual way. Note that there are no constants in the signature!

$$\exists x\, \exists y_1 \exists y_2 \big(y_1 \times y_1 = x \wedge y_2 \times y_2 = x \wedge y_1 \leq y_2 \wedge \neg(y_2 \leq y_1)\big)$$

There is a number with two distinct square-roots.
True in $(\mathbb{R}, +, \times, \leq)$. False in $(\mathbb{R}^+, +, \times, \leq)$

(d) Write a FO sentence that is true over $(\mathbb{R}, +, \times, \leq)$ but not true over $(\mathbb{Q}, +, \times, \leq)$. Note that there are no constants in the signature!

$$\exists x\, \exists y \exists z \big(x + x = x \wedge y \times y = y \wedge x \leq y \wedge \neg(y \leq x) \wedge z \times z = y + y \big)$$

$x + x = x$ says $x = 0$

$y \times y = y$ is satisfied only when $y = 0$ or $y = 1$

$x \leq y \wedge \neg(y \leq x)$ further restricts y to be 1.

$z \times z = y + y$ is insisting $z^2 = 2$

Which does not have a solution over rationals but does have a solution over reals.

3. **Modeling in propositional logic** [10 points]

Here is a puzzle.

A group of professors and politicians gather in a room. The professors always speak the truth and the politicians always lie.

There are three (distinct) people Ms. Scarlett, Mrs. Peacock, and Col. Mustard[1], and this is what they have to say about themselves.

Ms. Scarlett says: "All three of us are politicians."
Mrs. Peacock says:: "Exactly one of us three is a professor."

The puzzle is to figure out which of these are professors and which are politicians.

(a) Model the above using propositional logic.

(b) Find a satisfying assignment (you don't have to formally derive an assignment; guessing is fine) and write it clearly to show a solution for the puzzle.

Let Sp, Pp and Mp denote the statement "Ms. Scarlett is a professor", "Mrs. Peacock is a professor" and "Col. Mustard is a professor".

We can now model what Ms. Scarlett said as:

$$Sp \Leftrightarrow (\neg Sp \wedge \neg Pp \wedge \neg Mp)$$

(since professors speak truth always, what she says is true iff she is a professor)

We can model what Mrs. Peacock said as:

$$Pp \Leftrightarrow ((Sp \wedge \neg Pp \wedge \neg Mp) \vee (\neg Sp \wedge Pp \wedge \neg Mp) \vee (\neg Sp \wedge \neg Pp \wedge Mp))$$

(since professors speak truth always, what she says is true iff she is a professor)

The valuation $Sp : false$, $Pf : true$ and $Mp : false$ satisfies the constraints.

(Aside: It's also easy to see that Sp is forced to be false by the first constraint. And that if Pp is false, then Mp is true (by the first constraint), and hence by the second constraint Pp must be true, which is a contradiction. So Pp must be true and hence, by the second constraint, Mp must be false. So this is the only satisfying valuation.)

So Ms. Scarlet and Col. Mustard are politicians and Mrs. Peacock is a professor.

[1]All from Clue Jr., a game for kids

4. **Resolution** [14 points]

Prove the following is valid using resolution:

$$\varphi: \; ((p \Rightarrow (q \wedge r)) \wedge (q \Rightarrow s)) \Rightarrow (p \Rightarrow (r \wedge s))$$

$$\neg\varphi \equiv (p \Rightarrow (q \wedge r)) \wedge q \Rightarrow s \wedge \neg(p \Rightarrow (r \wedge s))$$

Writing in CNF:

$$\neg\varphi \equiv (\neg p \vee (q \wedge r)) \wedge (\neg q \vee s) \wedge (p \wedge \neg(r \wedge s))$$

$$\equiv (\neg p \vee q) \wedge (\neg p \vee r) \wedge (\neg q \vee s) \wedge p \wedge (\neg r \vee \neg s)$$

Writing as a set of clauses and doing resolution:

$\{\neg p, q\}$ $\{\neg p, r\}$ $\{\neg q, s\}$ $\{p\}$ $\{\neg r, \neg s\}$

$\{\neg p, s\}$ $\{r\}$

$\{s\}$ $\{\neg s\}$

$\perp$

5. **Inexpressiveness of FOL** [18 points]

Fix a first order signature with a single binary relation E. You can assume E is symmetric, i.e., $\forall x \forall y. E(x, y) \leftrightarrow E(x, y)$ holds. Hence the models can be seen as (finite or infinite) undirected graphs.

(a) Prove that there exists a set of FOL sentences X such that X holds precisely in the class of graphs that do not have a cycle.

(b) Prove that there does not exist a set of FOL sentences X such that X holds precisely in the class of graphs that have a cycle.

A graph has a cycle if there is a set of distinct nodes $u_1, \ldots, u_n$, for some $n \in \mathbb{N}$, $n \geq 3$ such that for every $1 \leq i < n$, $E(u_i, u_{i+1})$ holds, and $E(u_n, u_1)$ also holds.

Solution:

(a) Let φ_n, for any $n \geq 3$ be the sentence

$$\neg \exists x_1 \exists x_2 \ldots x_n \left(\bigwedge_{i,j \leq n, i \neq j} x_i = x_j \quad \wedge \quad \bigwedge_{i<n} E(x_i, x_{i+1} \quad \wedge \quad E(x_n, x_1) \right)$$

φ_n expresses that there is no cycle of length n.

Now consider $X = \{\varphi_n \mid n \in \mathbb{N}, n \geq 3\}$. Then, clearly, a model satisfies X iff there is no cycle on length n for any $n \geq$, i.e., there is no cycle.

(b) Proof by contradiction. Assume that there is a set X of sentences such that the models of X are precisely those that have a cycle.

Let φ_n, for any $n \geq 3$ be as above in part (a).
Consider $Z = X \cup \{\varphi_n \mid n \in \mathbb{N}, n \geq 3\}$.
Now we claim that every finite subset of X is satisfiable:

- Proof: Take any finite subset Y of the set Z. Then Y has only finitely many formulas of the kind φ_n.
  Let r be the largest number such that φ_r is in Y (if there is no such formula in Y, choose $r = 3$).
  Now the model consisting of $r+1$ elements and where the edge relation forms precisely a cycle on these elements. This is a model for X (since it has a cycle) and hence satisfies $Y \cap X$ as well. Also, it satisfies all formulas of the form φ_n in Y (as there are no cycles of length n, since $n \leq r$). Hence the model satisfies Y.

Since every finite subset of X is satisfiable, Z is satisfiable. Let M be a model satisfying all formulas in Z. Then M has a cycle (since it satisfies X) and at the same time has no cycle (since it satisfies all formulas φ_n, $n > 3$), which is a contradiction.

6. **Models** [10 points]

Consider the signature that consists of a constant symbol c and a unary function s.

Consider the sentence φ

$$\forall x.(\neg(s(x) = c)) \;\wedge\; \forall y \forall z.(s(y) = s(z) \Rightarrow y = z)$$

(a) Prove formally that *all* models of φ are infinite (preferably using induction).

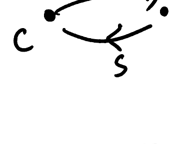
(b) Consider the formula above with only the second conjunct:

$$\forall y \forall z.(s(y) = s(z) \Rightarrow y = z)$$

Exhibit a finite model for it.

a) Let M be a model such that $M \models \varphi$.
Let u be the element interpreted to be c.
We claim that u, $s(u)$, $s(s(u))$, ... are all distinct,
which then shows M is an infinite model.
Proof by induction on n that $u, s(u), \ldots, s^n(u)$ are distinct.
Base case: $n=0$. Trivial as list has only one element, u.
Induction step: $n \geq 1$ Assume $u, s(u), \ldots, s^{n-1}(u)$ are distinct.
$s^n(u) \neq u$ since $M \models \forall x \ \neg(s(x)=c)$
By way of contradiction, assume $s^n(u) = s^m(u)$ $\quad m \neq n$, $m < n$, $m > 0$
Then $s(s^{n-1}(u)) = s(s^{m-1}(u))$
Since $M \models \varphi$, $s^{n-1}(u) = s^{m-1}(u)$ which contradicts our assumption. Hence $u, s(u), \ldots, s^n(u)$ are distinct. $\square$

b) Here is a finite model:



7. **Decidability** [12 points]

Let Σ be a finite signature (finitely many function symbols, relation symbols, and constant symbols).

Let Γ be a set of sentences such that for any $\alpha \in \Gamma$, α has a model if and only if α has a *finite* model (a model with a finite universe). You can also assume that Γ is recursive.

Consider the problem of deciding, given α in Γ, whether α is satisfiable.

Prove that the problem is decidable.

If you are not able to prove this, prove at least some parts of it (showing it's recursively enumerable, or it's co-re, or giving a semi-decision procedure) for partial credit.

Hint: Consider the language and its complement.

Proof. We will show satisfiability and validity are r.e.
and hence satisfiability is decidable.
(we can run the procedures for checking if
α is sat and $\neg\alpha$ is valid in tandem
and one procedure will halt and say YES;
if first procedure halts and says YES, then α is satisfiable,
if second procedure halts and says YES, then α is not satisfiable.)

Satisfiability over models of Γ is r.e. as
we can enumerate all finite models by increasing size
and halt and say YES, if there is a model satisfying α.
If α is satisfiable, then it has a finite model
and hence our procedure will definitely halt and
say YES.

Also, since Γ is recursive, and
$\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$
where $\vdash$ is a complete proof system for FO (there is one)
(by Gödel's completeness theorem)
we can enumerate all proofs to see if φ is valid.
If φ is valid, this procedure will terminate, and so
validity under Γ is r.e. $\square$