First-order logic

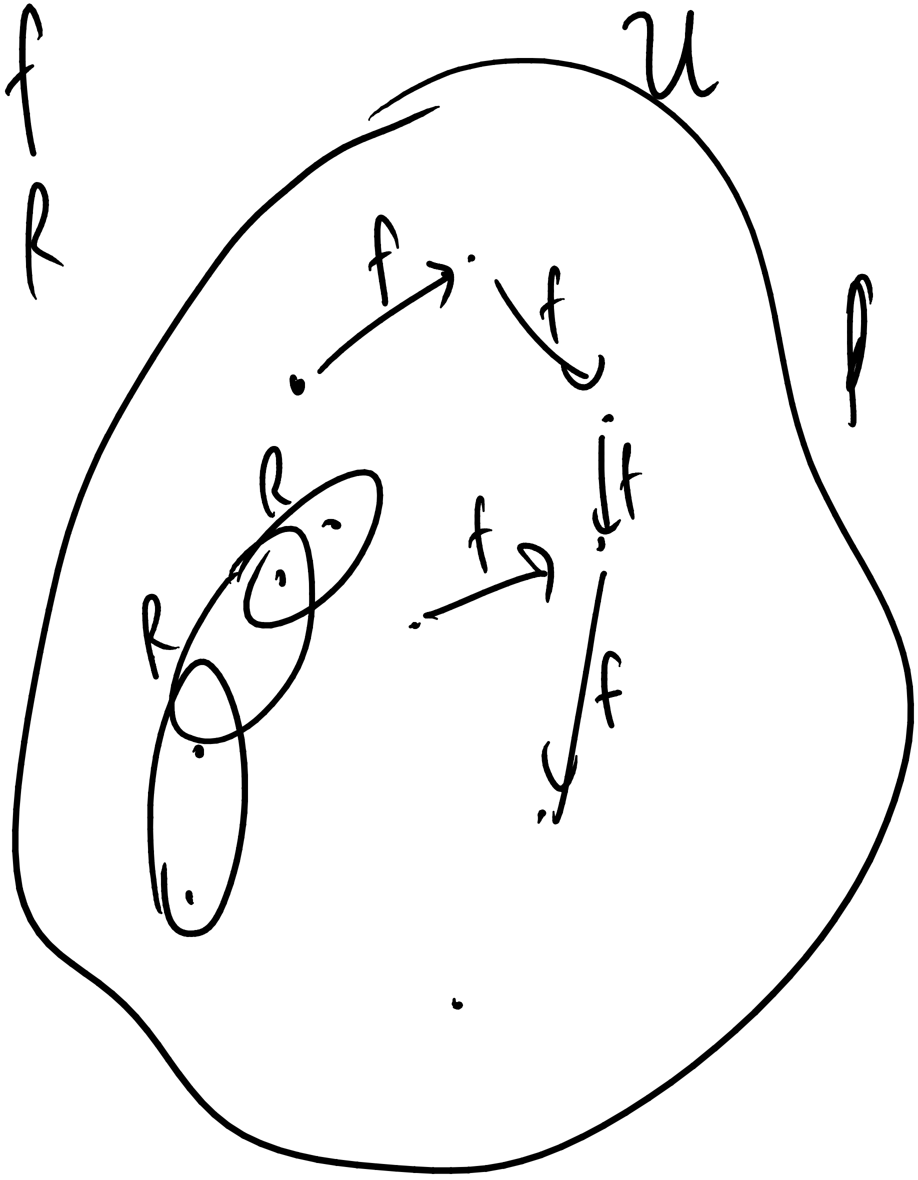
Numbers
Graphs
Linked-list
2 Sequences
Trees
Groups
Rings.

$$\forall n\ \exists m\ \left(m \overset{\text{relation}}{>} n \wedge \widetilde{\text{prime}}(m)\right)$$

$\forall$ forall

$\exists$ exists

$$\forall n\ n \neq n \underset{\text{function}}{+} 1$$


f
R
U
f
f
f
f
f
R
R

$\mathbb{N}$ ~~$\mathcal{U} = \{ \ldots -7, -6, \ldots$~~

$\mathcal{U} = \{0, 1, 2, 3, \ldots\}$

functions $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ $- : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$-_u : \mathbb{N} \to \mathbb{N}$ $*$

$\gcd : \mathbb{N} \to \mathbb{N}$

relations $< \subseteq \mathbb{N} \times \mathbb{N}$

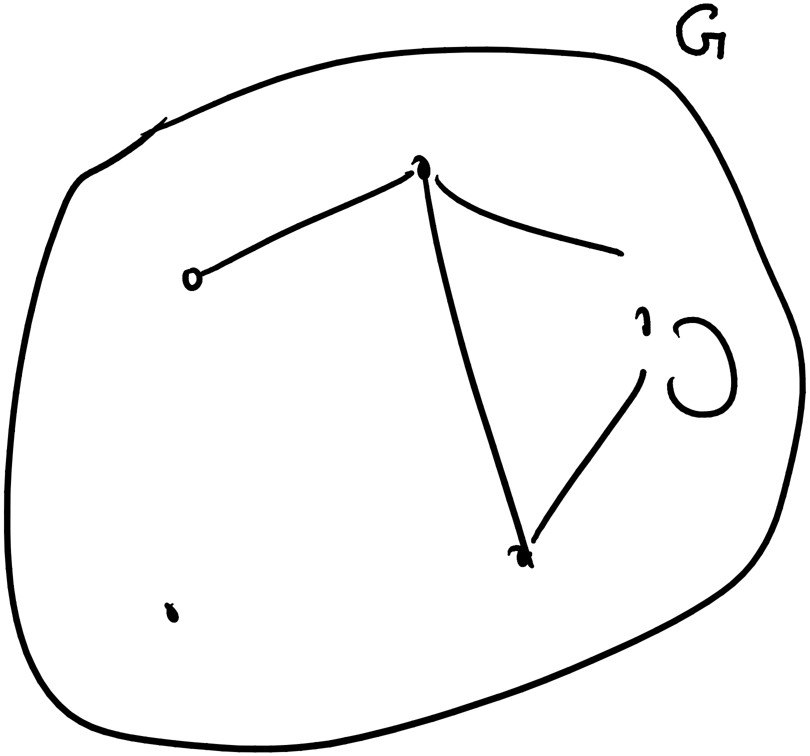
$| \subseteq \mathbb{N} \times \mathbb{N}$ $a|b$

Constants : $0, 1, 2$

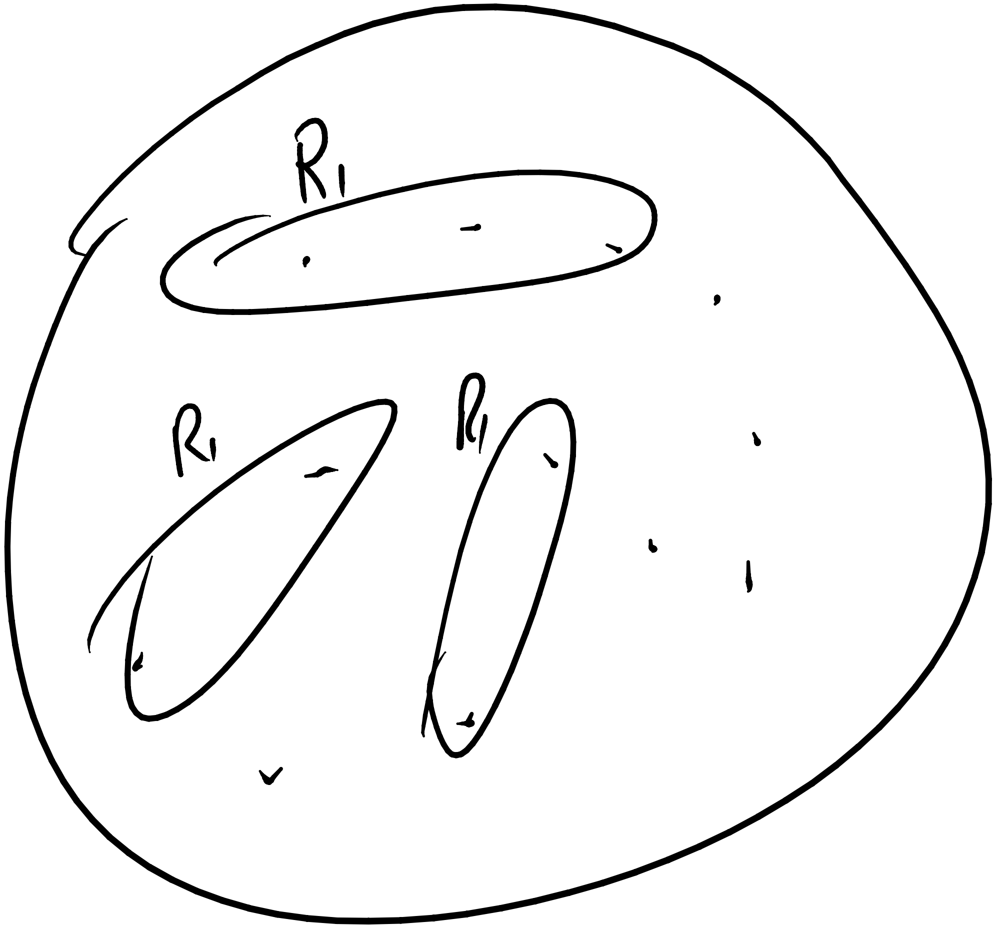
Graphs

$\mathcal{U}$: the set of nodes

Edge : $\mathcal{U} \times \mathcal{U}$.



Relational database

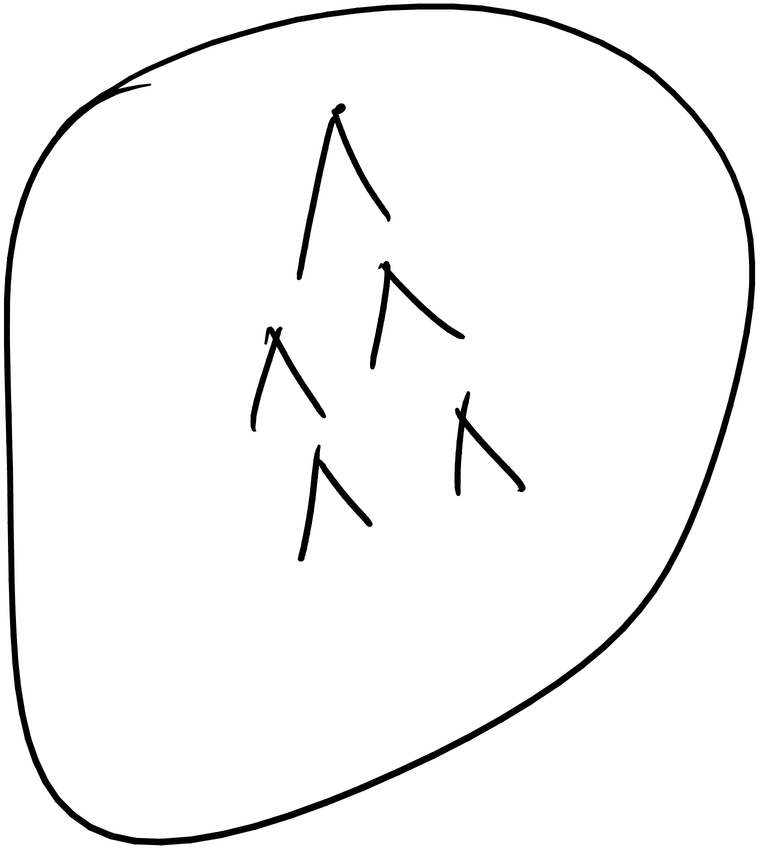


Trees

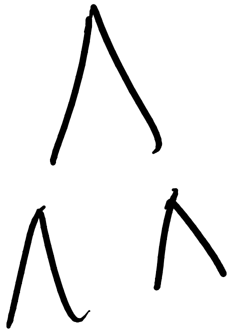
U : nodes

leftchild : $U \to U$

rightchild : $U \to U$



Programs



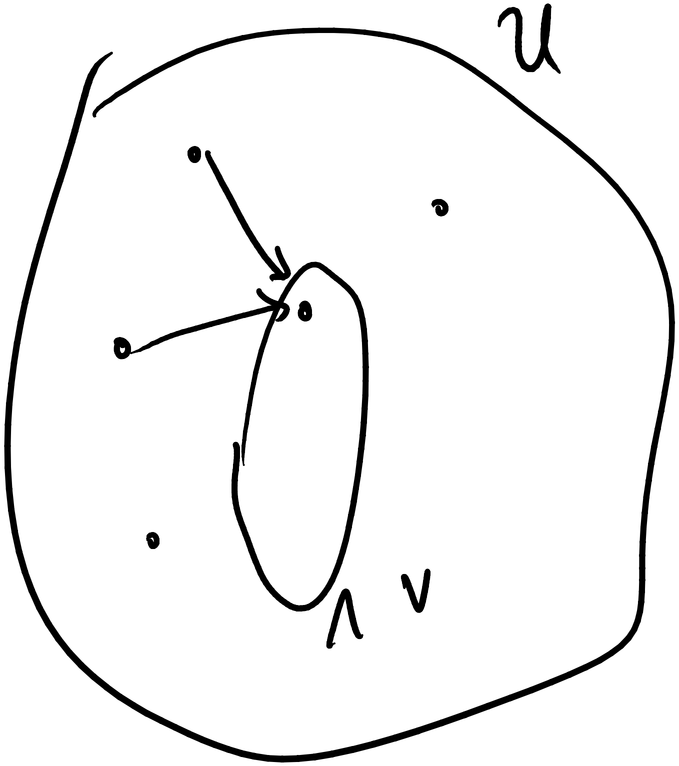
FOL

$$9 + 12 > 2$$

$$\vee \quad 2 > 5$$

$$\underbrace{R_>(\underbrace{f_+(\overset{term}{9},\overset{term}{12})}_{term},\overset{term}{2})}_{Formula} \vee \overset{Formula}{\underline{R_>(2,5)}}$$

$$\underline{\underline{\exists}}x.\ \underbrace{\overset{term}{x} > \overset{term}{5}}_{Formula}$$



FOL

$\mathcal{M} = (\mathcal{U}, f_1 \ldots f_k, R_1, \ldots R_n, c_1, \ldots c_m)$

Terms $t_1 \ldots t$: $c_i \mid f(t_1, \ldots t_r) \mid x$

Formulas α, β : $R(t_1 \ldots t_s) \mid \alpha \vee \beta \mid \neg\alpha \mid \exists x\, \alpha$

FOL $\equiv$ Formulas generated in the above grammar

$x > 5$

$\forall x\, \varphi \equiv \neg \exists x\, \neg\varphi$

$\mathcal{F} = \{ f_1 \ldots \}$ function symbols $\mathcal{R} = \{ R_1, R_2, \ldots \}$ relation symbols $\mathcal{C} = \{ c_1, c_2 \ldots \}$ constant symbols

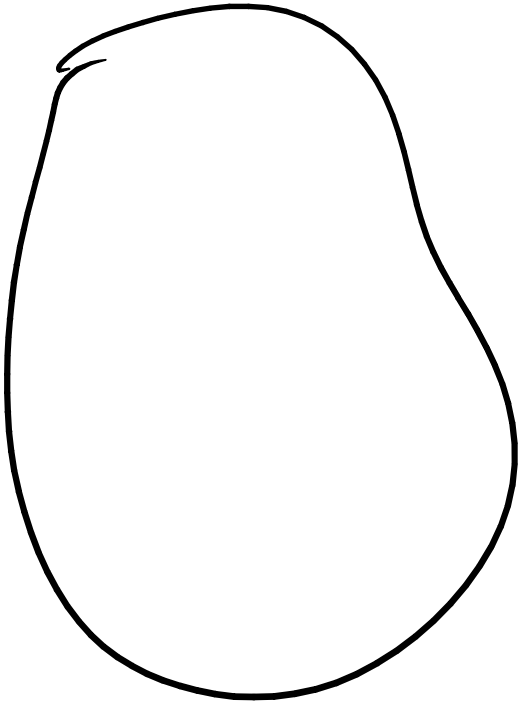
The graph is 3-colorable

$\exists \underline{\underline{S_1}}\ \underline{\underline{S_2}}\ \underline{\underline{S_3}}$

$\forall x .\ x \in S_1 \Rightarrow x \notin S_2$

$\vdots$

Sets $\equiv$ monadic relation $R(\cdot)$



$* \quad + \quad 0, \quad 1 \qquad \mathbb{N}$

$\leq \quad < \quad > \quad \geq \quad =$

prime(x) : $x > 1 \wedge \forall y ((y > 1 \wedge y < x)$

$\Rightarrow \neg \exists z . (y * z = x)))$

∞-many primes: $\forall x \; \exists y \, (y > x \wedge \text{prime}(y))$

Graph $\mathcal{U}$ - nodes E - edge relation

C_0, C_1, C_2, C_4 - unary relations

disjoint sets

partition of $\mathcal{U}$

It's a coloring:

$\neg$ ~~$\exists x \exists y$~~ $(-$

$$\forall x \forall y \left(E(x,y) \Rightarrow \left[\begin{array}{l} C_0(x) \Rightarrow \neg C_0(y) \\ \wedge \\ C_1(x) \Rightarrow \neg C_1(y) \\ \wedge \\ \vdots \\ C_4(x) \Rightarrow \neg C_4(y) \end{array} \right] \right) -$$

Semantics

$\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$

$ar : \mathcal{R} \to \mathbb{N}$

$ \mathcal{F} \to \mathbb{N}$

First order structure / model.

$$\mathcal{M} = (S, i)$$

S - universe ; nonempty

$$i : \begin{array}{l} f \in \mathcal{F} \longmapsto f^* : S^n \to S \\ R \in \mathcal{R} \longmapsto R^* \subseteq S^n \\ c \in \mathcal{C} \longmapsto c^* \in S \end{array}$$

x ✓ $\quad x \geq 5$ $\qquad\qquad$ $\underline{\underline{\exists x \; x \geq 5}}$ $\quad \mathcal{M}$

<u>Interpretation</u> $\mathcal{I} = (\underline{\underline{\mathcal{M}, \sigma}})$

$\mathcal{M}$ - fo structure

$\sigma : Var \to S$

Var : a countable infinite set of symbols $x_1, x_2 \ldots$

$\sigma[x \to s_1]$ Means the interpretation σ' that behaves exactly like σ except it maps x to s_1

$$\sigma[x \mapsto s_1] = \sigma'$$

where $\sigma'(x) = s_1$

$\sigma'(y) = \sigma(y)$ ~~$\forall y \neq x$~~

$\forall y \neq x$

$$\sigma[x_1 \mapsto s_1, x_2 \mapsto s_2 \ldots x_k \mapsto s_k]$$

$\mathcal{I} = (\mathcal{M}, \sigma)$ $\mathcal{M} = (S, i)$

$t^{\mathcal{I}}$: a particular element in S

$c^{\mathcal{I}} = \mathcal{M}(c)$

$x^{\mathcal{I}} = \sigma(x)$

$(f(t_1, \ldots t_n))^{\mathcal{I}} = f^{\mathcal{M}}(t_1^{\mathcal{I}}, \ldots t_n^{\mathcal{I}})$

$\mathcal{I} \models \alpha$ if "α holds in I" $=$

$I \models t_1 = t_2$ iff $t_1^I = t_2^I$

$I \models R(t_1, \ldots t_n)$ iff $R^{I}(t_1^I, \ldots t_n^I)$

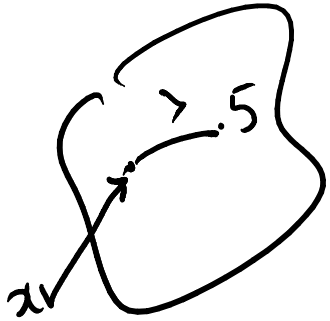
$I \models \neg \varphi$ iff $I \not\models \varphi$

$I \models \alpha \vee \beta$ iff $I \models \alpha$ or $I \models \beta$

$I \models \exists x \varphi$ iff there is an $s \in S$ such that $I[x \mapsto s] \models \varphi$

φ is satisfiable if there is an interpretation I (with a model M) such that $I \models \varphi$

$x > 5$



φ is valid if for every interpretation I $I \models \varphi$

$x > 5 \lor \neg(x > 5)$

φ $\qquad\qquad\qquad$ P_{φ}

$\underbrace{r(x,x')} \wedge \neg \underbrace{r(x,x')}$ $\qquad$ $P_{r(x,x')} \wedge \neg P_{r(x,x')}$

$x = y \wedge y = z \wedge \cancel{x \neq z}$
$\neg(x = z)$

$\Big\{ P_{x=y} \wedge P_{y=z} \wedge \neg P_{x=z}$
$\cancel{\wedge}, \left(P_{x=y} \wedge P_{y=z} \Rightarrow P_{x=z} \right) \Big\}$

$$\forall x \ r(x,t_2) \wedge \neg \exists x \ r(x,t_2)$$

$$\neg \underbrace{\exists x \ \neg r(x,t_2)}_{p} \wedge \neg \underbrace{\exists x \ r(x,t_2)}_{q}$$

$$\exists x \ r(x,t_2) \Rightarrow r(c,t_2)$$

C

$$r(\cancel{x}c,t_2) \Rightarrow \cancel{\neg \forall x \ r(x,t_2)}$$

$$\exists x \ r(x,t_2)$$

$$\underbrace{r(c,t_1)}_{u}$$

$$q \Rightarrow \cancel{p}u$$

$$u \Rightarrow q$$

Prime formulas (P_L)

A prime formula over $\overset{\text{Sig}}{L}$ is an atomic formula or a formula of the form $\exists x \psi$.

$$\underbrace{\exists x \, \overbrace{r(x)}} \vee \underbrace{t_1 = t_2}$$

A formula (FO) φ is "propositionally satisfiable" if there's a valuation $v : P_L \to \{\top, \bot\}$ that makes φ evaluate to true.

Prop. Let I be an interpretation.
Then there is a valuation $v : P_L \to \{\top, \bot\}$
such that $I \models \varphi$ iff $v \models \varphi$

Cor Let X be a set of FOL formulas.
If X is FO-sat then X is prop sat.

$L = (R, F, C)$

- let $C_0 = \phi$ and $L_0 = L$
- Assume C_n and L_n have been defined

  $L_n = (R, F, C \cup C_0 \cup C_1 \ldots \cup C_n)$

  For each formulae $\varphi(x)$ in $\Phi_{L_n} \setminus \Phi_{L_{n-1}}$

  with exactly one free variable x,

  let $C_{\varphi(x)}$ be a new constant

  ~~Let C_{n+1} be the set of~~ such constants

  $C_H = \bigcup_{i \geq 0} C_i$ $\quad L_H = (R, F, C \cup C_H)$

Axioms to add

Φ_H : Henkin axioms $\exists x \, \varphi(x) \Rightarrow \varphi(c_{\varphi(x)})$

Φ_Q : Quantifier axioms $\varphi(t) \Rightarrow \exists x \, \varphi(x)$

where t is a closed term in L_H.

Φ

Equality axioms

Φ_{Eq}

$$t = t$$

$$t = u \Rightarrow u = t$$

$$(t = u \wedge u = v) \Rightarrow (t = v)$$

$$(t_1 = u_1 \wedge t_2 = u_2 \dots \wedge t_n = u_n) \Rightarrow f(t_1, \dots t_n) = f(u_1, \dots u_n)$$

$$(t_1 = u_1 \wedge t_2 = u_2 \wedge \dots t_n = u_n) \Rightarrow \left[R(t_1 .. t_n) \Leftrightarrow R(u_1 \dots u_n) \right]$$

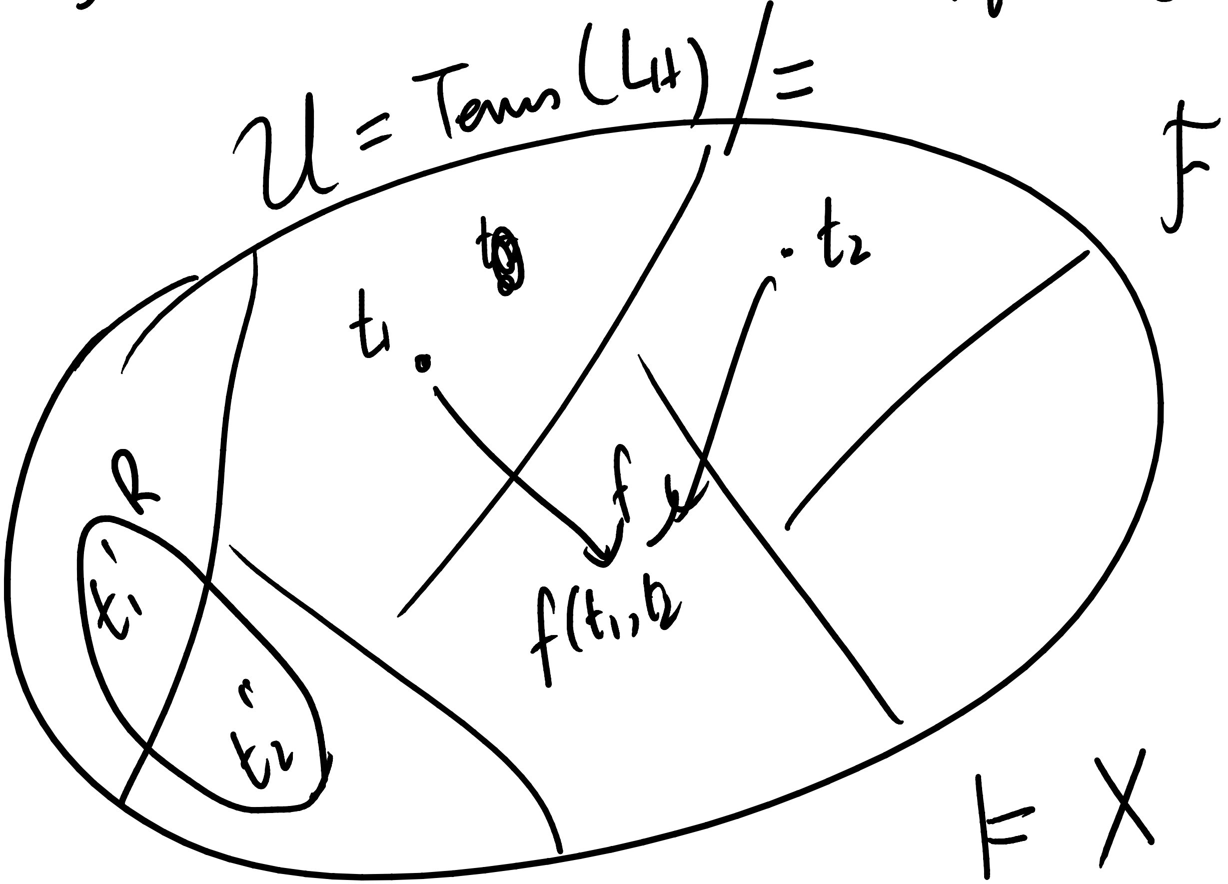
t, u, v are terms over L_H

Lemma (FO-Sat)

Fix L. Let L_H be its expansion. For any set X of formulas over L, then following are equivalent:

- 1) There is an $\mathcal{L}$-interpretation $I = (M, \sigma)$ which is a model for X
- 2) There is an $\mathcal{L}_H$-interpretation I' which is a model for X
- 3) $X \cup \Phi_H \cup \Phi_Q \cup \Phi_{Eq}$ is propositionally satisfiable

(3 ⇒ 2) Given a valuation v of prime formulas s.t. $v \models X \cup \Phi_H \cup \Phi_{Eq} \cup \Phi_Q$



$$L_H = (\mathcal{R}, \mathcal{F}, C \cup C_H)$$

$\tilde{I}_v$

$$\mathcal{U} = \{ [t] \mid t \text{ is a term over } L_H \}$$

For any $R \in \mathcal{R}$

$$R^M = \{([t_1],[t_2],\ldots[t_n]) \mid v \models R(t_1,\ldots t_n)\}$$

For any $f \in \mathcal{F}$

$$f^M([t_1],\ldots[t_n]) = [f(t_1,\ldots t_n)]$$

For any $c \in C \cup C_H$, $\quad c^M = [c]$

$x \in Var \quad \sigma(x) = [x]$

Indu on φ [illegible] $V \models$ "φ" iff $\underline{\underline{I_V \models \varphi}}$

because of Φ/Ω

$=$ ✓

$R(t_1, \ldots t_n)$ ✓

$\neg \varphi$ $\varphi \vee \varphi'$

$\varphi : \exists x\, \psi(x)$

$(\Leftarrow)$ $I_V \models \exists x\, \psi(x)$. Then there is an $s \in U$ s.t. $\psi(x)$ holds when ~~$x \mapsto$~~ s

$x \mapsto s$

But since my universe is terms, s corresponds to a term t.

$I_V \models \psi(t)$ by IH $V \models$ "$\psi(t)$" so $V \models \exists x \psi(x)$

$(\Rightarrow)$ $\quad v \models$ "$\exists x\, \psi(x)$"

$v \models$ "$\psi(c_\psi)$" $\quad$ by Φ_H

By IH $\quad I_v \models \psi(c_\psi)$

So $\quad I_v \models \exists x\, \psi(x)$.

Cor Löwenheim-Skolem theorem

If $\mathcal{L}$ ↗ sig is countable and
X is a set of formulas that
has ~~an infinite~~ a model,
then X has a countable ~~model~~
or finite model.

Compactness theorem

Let X be a set of FOL formulae.
Then X is satisfiable iff every $Y \subseteq_{fin} X$ is sat

$\Rightarrow$ easy

$\Leftarrow$ Let every finite subset of X be satisfiable.

I need to show X is satisfiable

i.e. $X \cup \Phi_H \cup \Phi_Q \cup \Phi_{Eq}$ is prop satisfiable

i.e. every finite subset of $X \cup \Phi_H \cup \Phi_Q \cup \Phi_{Eq}$ ~~is sat~~ is prop. sat. ~~is sat~~.

(by prop compactness theorem)

Let $Z \subseteq_{fin} X \cup \Phi_{H} \cup \Phi_{Q} \cup \Phi_{Eq}$

$$Z \subseteq (Z \cap X) \cup \Phi_{H} \cup \Phi_{Q} \cup \Phi_{Eq}.$$

$Z \cap X$ is a finite subset of X, and hence $Z \cap X$ is satisfiable

So $(Z \cap X) \cup \Phi_{H} \cup \Phi_{Q} \cup \Phi_{Eq}$ is prop satisfiable

So Z is prop. satisfiable.

Th Axiomatizations

Can be an infinite set of formulae (but recursive)

Cyclic graphs

Natural numbers

FOL cannot capture finiteness of models

$\Phi_1 : \forall x, \cancel{y} \forall y \quad x = y$

$\Phi_{\geq 2} : \exists x \exists y \quad x \neq y$

$\Phi_{\leq 3} : \exists x \exists y \exists z \left(\forall w .\ w = x \lor w = y \lor w = z \right)$

What we want is a ~~finite for~~ set of finite formula for

$$\text{"} \bigvee_{n \geq 1} \Phi_{\leq n} \text{"}$$

Thm. Let X be a set of formulas that has arbitrarily large models (i.e. for every $n \in \mathbb{N}$ there is a model of X of at least n elements). Then X ~~is~~ has a countably infinite model.

Proof. $\varphi_{\geq n}$ expresses that there are at least n elements. $n \in \mathbb{N}$, $n \geq 2$

$$X \cup \{ \varphi_{\geq n} \mid n \in \mathbb{N}, n \geq 2 \}$$

Every finite subset of this set is satisfiable since X has arbitrarily large models

So by compactness theorem, the set is satisfiable.

Take a model satisfying the set.

It must be infinite.

So there is an infinite (countable) model for X.

Non-standard models of arithmetic (countable)

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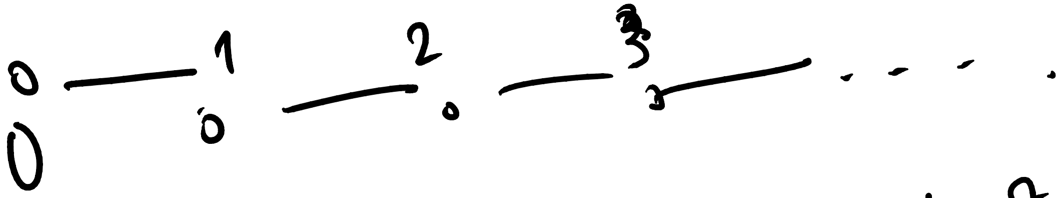


Assume that X is a set of formulas that hold for natural numbers.

$$X \cup \{x \neq 0,\ x \neq 1,\ x \neq 2, \dots\}$$

Skolem's theorem.

In fact $Th(\mathbb{N}) \cup \{x \neq 0, x \neq 1, \ldots\}$

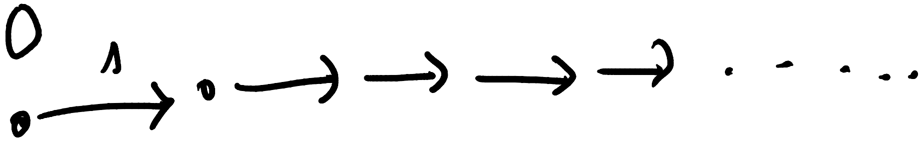
set of formulae
that hold in
the standard model
of nat numbers

is finitely satisfiable

Hence it is satisfiable

Hence there is a
countable non-standard
model for arithmetic.

(there is an element different
from 0, 1, 2, ...)



0 : constant

s : function unary

$\forall y, \forall z \quad s(y) = s(z) \Rightarrow y = z$

$\forall x \quad s(x) \neq 0$

Every element has at most one predecessor

0 has no predecessor

Undecidability

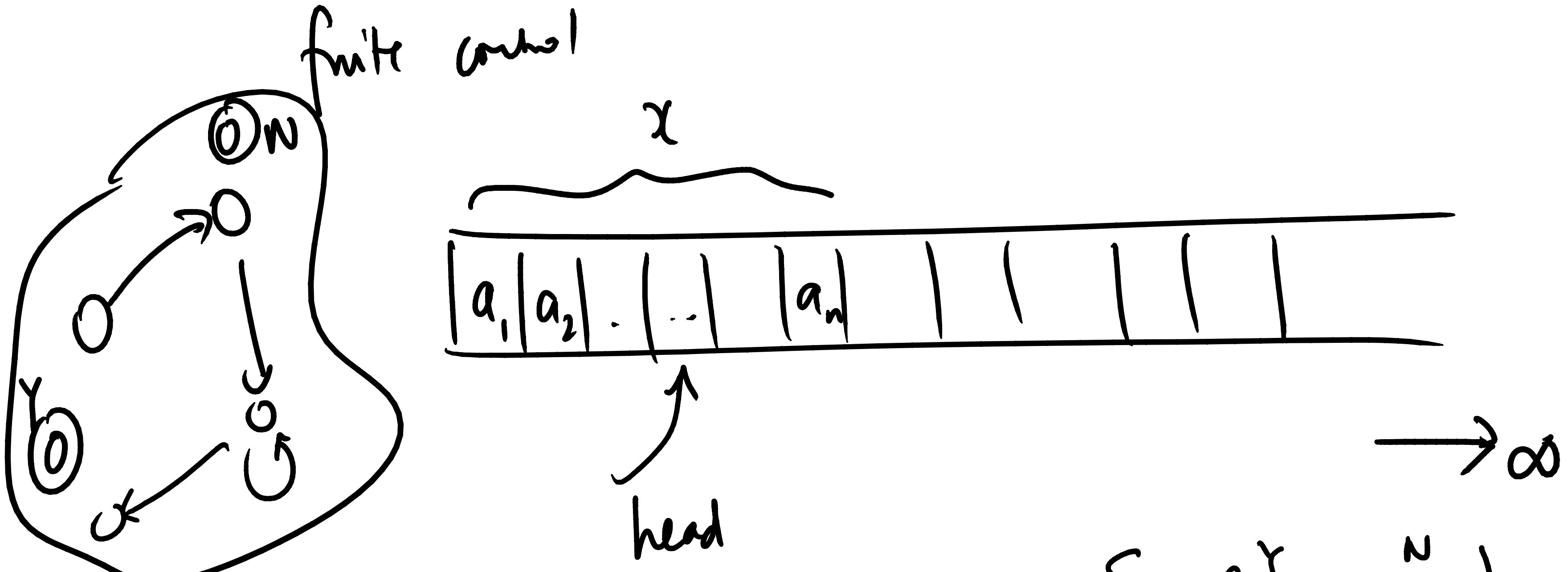
Σ finite

?

L ~~is~~ $\subseteq \Sigma^*$ is "computable" if there is a machine that, given any ~~$x \in \Sigma^*$~~, $x \in \Sigma^*$, can decide if $x \in L$.

It must always stop and give an answer.

Turing Machine



TM $M = (Q, q_0, \Sigma, \Gamma, \delta, q_{halt}^{Y}, q_{halt}^{N})$

finite set

$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$

Γ is finite

$\Sigma \subseteq \Gamma$

L is decidable if there is a TM that decides it.

$HALT_{TM} = \{ \langle TM \rangle \mid TM \text{ halts on empty tape} \}$

is not decidable.

Turing: $HALT_{TM}$ is not decidable.

Rice's theorem. Let $\mathcal{L}$ be a set of TMs defined by the property of the language accepted by TMs.

And let $\mathcal{L} \neq \emptyset$ nor $\mathcal{L}$ be all TMs

Then $\mathcal{L}$ is undecidable.

Recursively Enumerable

L is r.e. if there is a TM M s.t.

for every ~~$\forall$~~ $x \in L$, M halts and says "Y"

For every $x \notin L$, M either does not halt
or halts and says "N".

HALT_{TM} is r.e.

$\text{DOESNOTHALT}_{\text{TM}}$ is not. r.e.

If L is r.e. but not decidable

then $\overline{L}$ is not r.e.

Validity of FOL is r.e.

Satisfiability of FOL is not r.e.

Both are problems are undecidable

Undecidability of FOL

~~Hal~~ TM emptiness $\hookrightarrow$ FOL validity

M $\qquad\qquad$ φ_M

2CM

$\rightsquigarrow M = (Q, q_0, \Delta, F)$

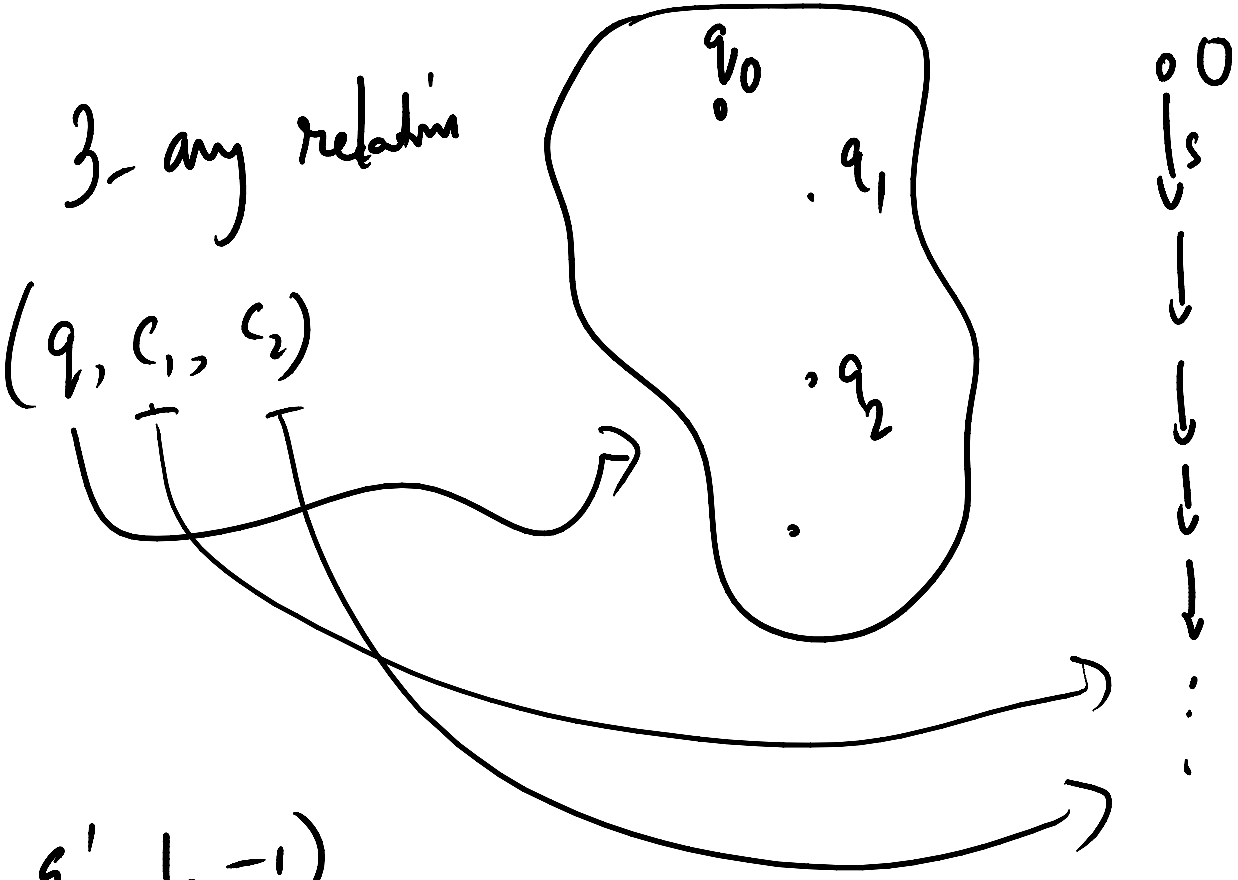
finite set of states

$q_0 \in Q$ initial state

final states

$$\Delta \subseteq Q \times \{0,1\} \times \{0,1\} \times Q \times \{-1,0,1\} \times \{-1,0,1\}$$

Conf : a 3-ary relation



$\underline{(q, 0, 1, q', 1, -1)}$

$$\forall x. \left(conf(q,0,x) \wedge \exists y\,(x=S(y)) \Rightarrow \exists y.\, conf(q', \cancel{x}\, S(0), y) \wedge \cancel{y}\, x=S(y) \right)$$

$$\underline{init} \triangleq conf(q_0, 0, 0)$$

$$\underline{final} \triangleq \exists x \, \exists y \bigvee_{q \in F} conf(q, x, y)$$

$$\underline{delta} \triangleq \bigwedge_{t \in \Delta} \varphi_t$$

$$\varphi_M : (\underline{init} \wedge \underline{delta}) \Rightarrow \underline{final}$$

Thm. Validity of FOL is undecidable
Satisfiability of FOL is undecidable.

Gödel's Completeness Theorem

There is a proof system that is (sound and) complete for FOL.

$$\models \varphi \quad \text{iff} \quad \vdash_{PS} \varphi$$

Hence Valid $= \{ \varphi \mid \models \varphi \}$ is r.e.

Since Valid is undecidable, Satisfiable $= \{ \varphi \mid \exists M.\ M \models \varphi \}$ is not r.e.

Strong Completeness Theorem

For any X, $X \vDash \varphi$ iff $X \vdash_{PS} \varphi$

For any model M that satisfies every formula in X, M also satisfies φ.

There is a proof of φ using axioms in PS and X.

Detour : Propositional logic

Theorem (Deduction Theorem)

Let X be a set of prop formulas.

And α, β be prop formulas

$$X \cup \{\alpha\} \vdash \beta \quad \text{iff} \quad X \vdash \alpha \Rightarrow \beta$$

Strong completeness for prop logic

$$X \models \alpha \quad \text{iff} \quad X \vdash_{PS} \alpha$$

($\Leftarrow$) Soundness ✓

($\Rightarrow$) Suppose $X \models \alpha$

Then $X \cup \{\neg\alpha\}$ is not satisfiable.

By compactness theorem, there is a finite subset that is unsat. (Y) So $Y \cup \{\neg\alpha\}$ is unsat.

So $Y \models \alpha$

So: let $Y = \{\beta_1, \ldots \beta_m\}$

Then $\models \beta_1 \Rightarrow \beta_2 \Rightarrow \ldots \beta_m \Rightarrow \alpha$

By completeness theorem, $\vdash \beta_1 \Rightarrow \beta_2 \Rightarrow \ldots \beta_m \Rightarrow \alpha$

By deduction theorem.

$\{\beta_1\} \vdash \beta_2 \Rightarrow \ldots \beta_m \Rightarrow \alpha$

$\vdots$

$\{\beta_1, \ldots \beta_m\} \vdash \alpha$ So $X \vdash \alpha$

$\square$

Back to FOL

A complete proof system for FOL:

A complete proof system of prop logic

(A1) All tautologies of prop logic

(A2a) $x = x$

(A2b) $t = u \Rightarrow \varphi(t) \Leftrightarrow \varphi(u)$ for any atomic formula φ

(A3) $\varphi(t) \Rightarrow \exists x\, \varphi(x)$

(MP) $$\frac{\varphi, \quad \varphi \Rightarrow \psi}{\psi}$$

(G) $$\frac{\varphi(x) \Rightarrow \psi}{[\exists x\, \varphi(x)] \Rightarrow \psi}$$

where ψ does not have x free

Lemma. All equality axioms can be derived in the proof system.

Lemma Let X be a set of formulas

i) If $X \vdash \varphi \Rightarrow \psi$ and $X \vdash \neg\varphi \Rightarrow \psi$ then $X \vdash \psi$

ii) If $X \vdash (\varphi \Rightarrow \theta) \Rightarrow \psi$ then $X \vdash \neg\varphi \Rightarrow \psi$ and $X \vdash \theta \Rightarrow \psi$

iii)

iii) If $x \notin FV(\psi)$ and $X \vdash \left[\left(\exists y\, \varphi(y) \Rightarrow \varphi(x) \right) \cancel{\Rightarrow \psi} \Rightarrow \psi \right]$

then $X \vdash \psi$

Completeness (Strong) $X \models \varphi$ then $X \vdash \varphi$

Let $X \models \varphi$. So $X \cup \{\neg\varphi\}$ is not sat

~~Then $X \cup \{\neg\varphi\}$ is not satisfiable ✓~~

~~So there is a finite subset~~

$$X \cup \{\neg\varphi\} \cup \Phi_H \cup \Phi_Q \cup \Phi_{Eq}$$ is not prop satisfiable

By compactness of prop logic, there's a finite subset Y

$Y \cup \{\neg\varphi\}$ is not satisfiable

Let $\alpha_1 \ldots \alpha_n$ be ~~[illegible]~~ members of γ that belong to $X \cup \Phi_Q \cup \Phi_{Eq}$

Let $\beta_1 \ldots \beta_m$ be the members of γ that belong to Φ_H.

Rank (β_i) is the level at which the constants in β were first introduced.

Arrange $\beta_1 \ldots \beta_m$ s.t. $\text{rank}(\beta_i) \geq \text{rank}(\beta_{i+1})$

Since $Y \cup \{\neg \varphi\}$ is not prop sat,

$$\alpha_1 \Rightarrow (\alpha_2 \Rightarrow \ldots (\alpha_n \Rightarrow (\beta_1 \Rightarrow \ldots (\beta_m \Rightarrow \varphi))\ldots)$$

is ~~valid~~ a tautology.

So $X \vdash \alpha_1 \Rightarrow \alpha_2 \Rightarrow . \beta_1 \Rightarrow .. \beta_m \Rightarrow \varphi$ [replace every new constant with a new variable]

Since I can prove each α_i

$X \vdash \beta_1 \Rightarrow \beta_2 \Rightarrow .. \beta_m \Rightarrow \varphi$

β_i' $\exists x\, \psi(x) \Rightarrow \psi(y)$

$\vdash \left[\exists x\, \psi(x) \Rightarrow \psi(y)\right] \Rightarrow \alpha$

then $\vdash \alpha$ by Lemma (iii)

You can prove these assumptions $\beta_1 \ldots \beta_m$ one by one.

So $X \vdash \varphi$

Recap

FO syntax and semantics

$\models \varphi$ validity / tautology (φ is true in all models)

$X \models \varphi$ X entails φ
(In all models that sat every formula in X, φ also holds)

$M \models \varphi$ φ holds in model M

$\vdash \varphi$ φ is derivable in "the" proof system

$X \vdash \varphi$ Using axioms from X as well, ~~you~~ there is a derivation of φ

Compactness Thm:

If X is not satisfiable then there is a finite subset Y of X such that Y is not satisfiable.

If $X \models \varphi$ then ~~iff~~ there is $Y \subseteq_{fin} X$ s.t. $Y \models \varphi$

(If $X \models \varphi$, then $X \cup \{\neg\varphi\}$ is not sat

So there is a finite $Y \subseteq_{fin} X$ s.t. $Y \cup \{\neg\varphi\}$ is not sat

So $Y \models \varphi$)

Completeness Thm. (Strong)

$$X \models \varphi \quad \text{iff} \quad X \vdash \varphi$$

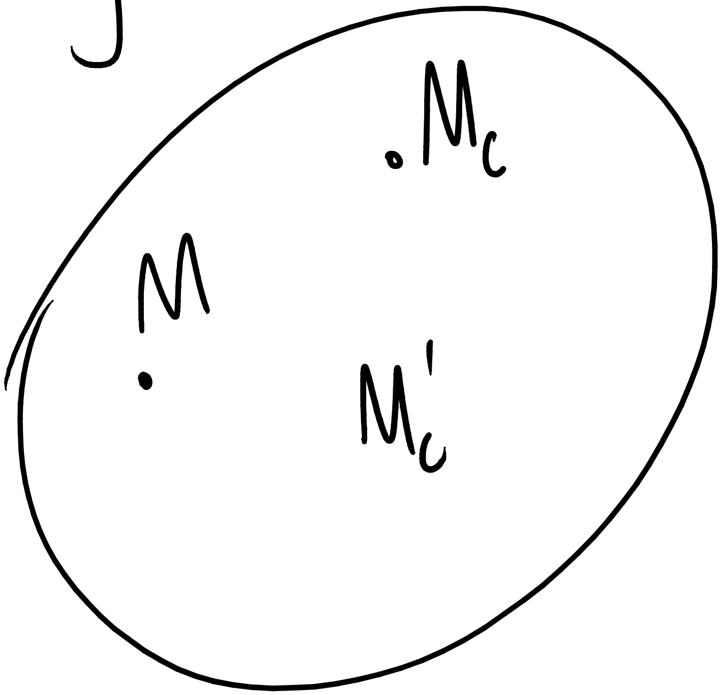
X can be infinite / finite / empty.

Theories

M

Unless M is finite, there is no set X of FO formulas such that M is the only model satisfying ~~for~~ X.

Is there a set X (finite/recursive) such that $X \models \varphi$ iff $M \models \varphi$



Recap : If X is recursive

~~can~~ then $\{\varphi \mid X \vdash \varphi\}$

is recursively enumerable.

recursive

Suppose $X \vDash \varphi$ iff $M \vDash \varphi$

$$\mathrm{Th}(M) = \{\varphi \mid M \vDash \varphi\} = \{\varphi \mid X \vDash \varphi\}$$

$$= \{\varphi \mid X \vdash \varphi\}$$

So $\mathrm{Th}(M)$ is rec. enumerable.

In fact $\mathrm{Th}(M)$ is <u>decidable</u>.

Proof

Either φ or $\neg\varphi$ belongs to $\mathrm{Th}(M)$

Search for proof of φ and $\neg\varphi$ ~~s~~ in parallel.

$(\mathbb{N}, +, 0, 1, <, 1)$

Is there a recursive X that captures this model up to FOL.

Yes! (Presburger arithmetic)

$$(\mathbb{N}, +, *, 0, 1, s \cancel{\}}, \leq)$$

No!

Gödel's first incompleteness theorem

There is no recursive set X of FO formulas s.t. $X \vdash \varphi$ iff φ holds in $(\mathbb{N}, +, *)$

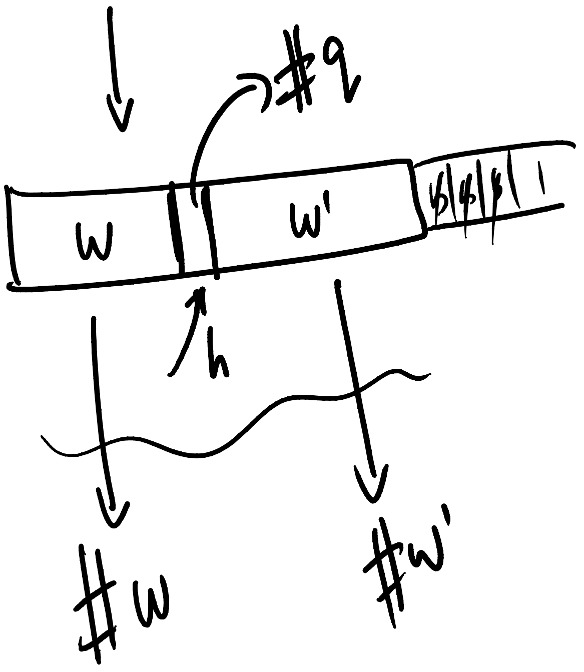
If there was such an X then $Th(\mathbb{N}, +, *)$ would be decidable

True theory of $(\mathbb{N}, +, *)$ is undecidable.

TM halting $\overset{\text{reduces to}}{\rightsquigarrow}$ ~~Ba~~ Satisfiability of φ in $(\mathbb{N}, +, *)$

$\langle C_1, \ldots \quad C_n \rangle \rightsquigarrow m$

$\#\langle \#C_1, \#C_2 \ldots \#C_n \rangle$



$M \rightsquigarrow \mathcal{C}$, a class of models

$$\mathrm{Th}(\mathcal{C}) = \{ \varphi \mid \text{for every model } M \text{ in } \mathcal{C},\ M \models \varphi \}$$

φ - partial orders $\leq$ $=$

$\exists x \exists y \neg(x \leq y)$

✗

~~$\exists x \exists y \neg(x \leq y)$~~

$\forall x \forall y\ x \leq y$ $\xcancel{\emptyset}$

φ

$(\forall x \forall y\ x \leq y) \Rightarrow \exists x \forall y\ x = y$

We would like axiomatizations X such that

$$X \models \varphi \quad \text{iff} \quad \varphi \in \mathrm{Th}(\mathcal{C})$$

It could be the case that $\mathcal{C}$ has a recursive axiomatization but $\mathrm{Th}(\mathcal{C})$ is <u>undecidable</u>.

(but of course it is r.e.)