Write your answers in the separate answer booklet.
Please return this question sheet and your cheat sheet with your answers.

1. Recall that a boolean formula is in *conjunctive normal form* if it is the conjunction (AND) of a series of *clauses*, each of which is a disjunction (OR) of a series of literals, each of which is either a variable or the negation of a variable. Consider the following variants of SAT:

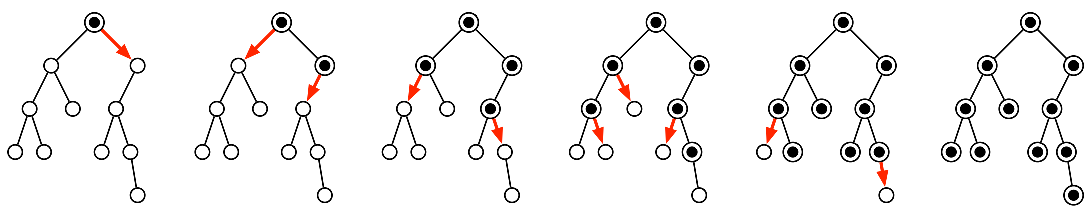
   - ***3SAT***: Given a boolean formula Φ in conjunctive normal form, such that every clause in Φ contains exactly *three* literals, is Φ satisfiable?
   - ***4SAT***: Given a boolean formula Φ in conjunctive normal form, such that every clause in Φ contains exactly *four* literals, is Φ satisfiable?

   (a) Describe a polynomial-time reduction from 3SAT to 4SAT.

   (b) Describe a polynomial-time reduction from 4SAT to 3SAT.

   Don't forget to ***prove*** that your reductions are correct!

2. Suppose we need to distribute a message to all the nodes in a given binary tree. Initially, only the root node knows the message. In a single round, each node that knows the message is allowed (but not required) forward it to at most one of its children. Describe and analyze an algorithm to compute the minimum number of rounds required for the message to be delivered to all nodes in the tree.



A message being distributed through a binary tree in five rounds.

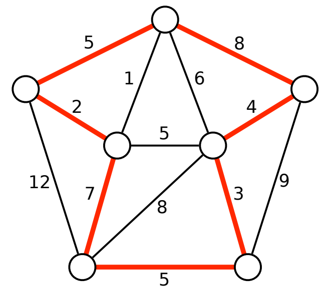
3. The ***maximum acyclic subgraph*** problem is defined as follows: The input is a directed graph $G = (V, E)$ with n vertices. Our task is to label the vertices from 1 to n so that the number of edges $u \to v$ with $label(u) < label(v)$ is as large as possible. Solving this problem exactly is NP-hard.

   (a) Describe and analyze an efficient 2-approximation algorithm for this problem.

   (b) ***Prove*** that the approximation ratio of your algorithm is at most 2.

   [Hint: Find an ordering of the vertices such that at least half of the edges point forward. Why is that enough?]

4. Let G be an undirected graph with weighted edges. A *heavy Hamiltonian cycle* is a cycle C that passes through each vertex of G exactly once, such that the total weight of the edges in C is at least half of the total weight of all edges in G. ***Prove*** that deciding whether a graph has a heavy Hamiltonian cycle is NP-hard.



A heavy Hamiltonian cycle. The cycle has total weight 34; the graph has total weight 67.

5. Lenny Rutenbar, founding dean of the new Maximilian Q. Levchin College of Computer Science, has commissioned a series of snow ramps on the south slope of the Orchard Downs sledding hill[1] and challenged William (Bill) Sanders, head of the Department of Electrical and Computer Engineering, to a sledding contest. Bill and Lenny will both sled down the hill, each trying to maximize their air time. The winner gets to expand their department/college into both Siebel Center and the new ECE Building; the loser has to move their entire department/college under the Boneyard bridge next to Everitt Lab.

   Whenever Lenny or Bill reaches a ramp *while on the ground*, they can either use that ramp to jump through the air, possibly flying over one or more ramps, or sled past that ramp and stay on the ground. Obviously, if someone flies over a ramp, they cannot use that ramp to extend their jump.

   Suppose you are given a pair of arrays $Ramp[1..n]$ and $Length[1..n]$, where $Ramp[i]$ is the distance from the top of the hill to the ith ramp, and $Length[i]$ is the distance that a sledder who takes the ith ramp will travel through the air. Describe and analyze an algorithm to determine the maximum total distance that Lenny or Bill can spend in the air.

[1]The north slope is faster, but too short for an interesting contest.