Probabilistic Counting and Morris Counter

Lecture 04
September 1, 2022
Part I

Counting Events
Streaming model

- The input consists of $m$ objects/items/tokens $e_1, e_2, \ldots, e_m$ that are seen one by one by the algorithm.
- The algorithm has “limited” memory say for $B$ tokens where $B < m$ (often $B \ll m$) and hence cannot store all the input.
- Want to compute interesting functions over input.
Counting problem

Simplest streaming question: how many events in the stream?

Obvious:
A counter that increments on seeing each new item.

Requires $\lceil \log n \rceil = \Theta(\log n)$ bits to be able to count up to $n$ events.

(We will use $n$ for length of stream for this lecture)

Question:
can we do better?

Not deterministically.

Yes, with randomization.

"Counting large numbers of events in small registers" by Robert Morris (Bell Labs), Communications of the ACM (CACM), 1978
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**Probabilistic Counting Algorithm**

**ProbabilisticCounting:**

\[ X \leftarrow 0 \]

While (a new event arrives)

Toss a biased coin that is heads with probability \( \frac{1}{2^X} \)

If (coin turns up heads)

\[ X \leftarrow X + 1 \]

endWhile

Output \( 2^X - 1 \) as the estimate for the length of the stream.
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**Theorem**

Let \( Y = 2^X \). Then \( E[Y] - 1 = n \), the number of events seen.
Morris’s motivation:

- Had 8 bit registers. Can count only up to $2^8 = 256$ events using deterministic counter. Had many counters for keeping track of different events and using 16 bits (2 registers) was infeasible.

- If only $\log \log n$ bits then can count to $2^{2^8} = 2^{256}$ events! In practice overhead due to error control etc. Morris reports counting up to 130,000 events using 8 bits while controlling error.

See 2 page paper for more details.
Induction on $n$. For $i \geq 0$, let $X_i$ be the counter value after $i$ events. Let $Y_i = 2^{X_i}$. Both are random variables.
Analysis of Expectation

Induction on $n$. For $i \geq 0$, let $X_i$ be the counter value after $i$ events. Let $Y_i = 2^{X_i}$. Both are random variables.

**Base case:** $n = 0, 1$ easy to check: $X_i, Y_i - 1$ deterministically equal to $0, 1$. 
Analysis of Expectation

\[ E[Y_n] = E[2^{X_n}] = \sum_{j=0}^{\infty} 2^j \Pr[X_n = j] \]

\[ = \sum_{j=0}^{\infty} 2^j \left( \Pr[X_{n-1} = j] \cdot \left(1 - \frac{1}{2^j}\right) + \Pr[X_{n-1} = j - 1] \cdot \frac{1}{2^{j-1}} \right) \]

\[ = \sum_{j=0}^{\infty} 2^j \Pr[X_{n-1} = j] \]

\[ + \sum_{j=0}^{\infty} (2 \Pr[X_{n-1} = j - 1] - \Pr[X_{n-1} = j]) \]

\[ = E[Y_{n-1}] + 1 \quad \text{(by applying induction)} \]

\[ = n + 1 \]
Jensen’s Inequality

**Definition**

A real-valued function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is convex if 
\[ f\left(\frac{a + b}{2}\right) \leq \frac{f(a) + f(b)}{2} \]
for all \( a, b \). Equivalently, 
\[ f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b) \]
for all \( \lambda \in [0, 1] \).
Jensen’s Inequality

**Definition**

A real-valued function \( f : \mathbb{R} \to \mathbb{R} \) is **convex** if

\[
f((a + b)/2) \leq (f(a) + f(b))/2 \quad \text{for all } a, b.
\]

Equivalently,

\[
f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b) \quad \text{for all } \lambda \in [0, 1].
\]

**Theorem (Jensen’s inequality)**

Let \( Z \) be random variable with \( \mathbb{E}[Z] < \infty \). If \( f \) is convex then

\[
f(\mathbb{E}[Z]) \leq \mathbb{E}[f(Z)].
\]
Implication for counter size

We have $Y_n = 2^{X_n}$. The function $f(z) = 2^z$ is convex. Hence

$$2^{E[X_n]} \leq E[Y_n] \leq n + 1$$

which implies

$$E[X_n] \leq \log(n + 1)$$

Hence expected number of bits in counter is $\lceil \log \log(n + 1) \rceil$. 
Question: Is the random variable $Y_n$ well behaved even though expectation is right? What is its variance? Is it concentrated around expectation?

Lemma

$$E[Y_n^2] = 3n^2 + 3n + 1$$
and hence

$$Var[Y_n] = \frac{n(n-1)}{2}$$
**Variance calculation**

**Question:** Is the random variable $Y_n$ well behaved even though expectation is right? What is its variance? Is it concentrated around expectation?

**Lemma**

\[
E[Y_n^2] = \frac{3}{2} n^2 + \frac{3}{2} n + 1 \quad \text{and hence} \quad \text{Var}[Y_n] = \frac{n(n - 1)}{2}.
\]
Variance analysis

Analyze $E[Y_n^2]$ via induction.

Base cases: $n = 0, 1$ are easy to verify since $Y_n$ is deterministic.

\[
E[Y_n^2] = E[2^{2X_n}] = \sum_{j \geq 0} 2^j \cdot \Pr[X_n = j]
\]

\[
= \sum_{j \geq 0} 2^j \cdot \left( \Pr[X_{n-1} = j](1 - \frac{1}{2^j}) + \Pr[X_{n-1} = j - 1] \frac{1}{2^{j-1}} \right)
\]

\[
= \sum_{j \geq 0} 2^j \cdot \Pr[X_{n-1} = j]
\]

\[
+ \sum_{j \geq 0} \left( -2^j \Pr[X_{n-1} = j - 1] + 42^{j-1} \Pr[X_{n-1} = j - 1] \right)
\]

\[
= E[Y_{n-1}^2] + 3E[Y_{n-1}]
\]

\[
= \frac{3}{2} (n - 1)^2 + \frac{3}{2} (n - 1) + 1 + 3n = \frac{3}{2} n^2 + \frac{3}{2} n + 1.
\]
Error analysis via Chebyshev inequality

We have $\mathbb{E}[Y_n] = n$ and $\text{Var}(Y_n) = n(n - 1)/2$ implies $\sigma_{Y_n} = \sqrt{n(n - 1)/2} \leq n$.

Applying Chebyshev’s inequality:

$$\Pr[|Y_n - \mathbb{E}[Y_n]| \geq tn] \leq 1/(2t^2).$$

Hence constant factor approximation with constant probability (for instance set $t = 1/2$).
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**Question:** Want estimate to be tighter. For any given \( \epsilon > 0 \) want estimate to have error at most \( \epsilon n \) with say constant probability or with probability at least \( (1 - \delta) \) for a given \( \delta > 0 \).
Part II
Improving Estimators
Probabilistic Estimation

**Setting:** want to compute some real-value function $f$ of a given input $I$

**Probabilistic estimator:** a randomized algorithm that given $I$ outputs a random answer $X$ such that $\mathbb{E}[X] \simeq f(I)$. Estimator is **exact** if $\mathbb{E}[X] = f(I)$ for all inputs $I$.

**Additive approximation:** $|\mathbb{E}[X] - f(I)| \leq \epsilon$

**Multiplicative approximation:** $(1 - \epsilon)f(I) \leq \mathbb{E}[X] \leq (1 + \epsilon)f(I)$
Probabilistic Estimation

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**Probabilistic estimator:** a randomized algorithm that given $I$ outputs a random answer $X$ such that $E[X] \simeq f(I)$. Estimator is **exact** if $E[X] = f(I)$ for all inputs $I$.

**Additive approximation:** $|E[X] - f(I)| \leq \epsilon$

**Multiplicative approximation:** $(1 - \epsilon)f(I) \leq E[X] \leq (1 + \epsilon)f(I)$

**Question:** Estimator only gives expectation. Bound on $\text{Var}[X]$ allows Chebyshev. Sometimes Chernoff applies. How do we improve estimator?
Variance reduction via averaging

- Run $h$ parallel copies of algorithm with independent randomness
- Let $Y^{(1)}, Y^{(2)}, \ldots, Y^{(h)}$ be estimators from the $h$ parallel copies
- Output $Z = \frac{1}{h} \sum_{i=1}^{h} Y^{(i)}$

Claim: $E[Z_n] = n$ and $\text{Var}(Z_n) = \frac{1}{h} \left( n \left( n - 1 \right) / 2 \right)$.

Choose $h = 2^{\epsilon^2}$. Then applying Chebyshev's inequality $\Pr[|Z_n - E[Z_n]| \geq \epsilon n] \leq \frac{1}{4}$.

To run $h$ copies need $O\left( \frac{1}{\epsilon^2} \log \log n \right)$ bits for the counters.
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Choose \( h = \frac{2}{\epsilon^2} \). Then applying Chebyshev’s inequality

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\Pr[|Z_n - E[Z_n]| \geq \epsilon n] \leq 1/4.
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Choose $h = \frac{2}{\epsilon^2}$. Then applying Chebyshev’s inequality

$$\Pr[|Z_n - E[Z_n]| \geq \epsilon n] \leq 1/4.$$ 

To run $h$ copies need $O\left(\frac{1}{\epsilon^2} \log \log n\right)$ bits for the counters.
Error reduction via median trick

We have:

$$\Pr[|Z_n - E[Z_n]| \geq \epsilon n] \leq 1/4.$$ 

Want:

$$\Pr[|Z_n - E[Z_n]| \geq \epsilon n] \leq \delta$$

for some given parameter $\delta$. 

Idea: Repeat independently $c \log(1/\delta)$ times for some constant $c$. Why?

Which one should we pick?

Algorithm: Output median of $Z(1)$, $Z(2)$, ..., $Z(\ell)$. 

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for some given parameter \( \delta \).

Can set \( h = \frac{1}{2\epsilon^2\delta} \) and apply Chebyshev. Better dependence on \( \delta \)?
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**Idea:** Repeat independently \( c \log(1/\delta) \) times for some constant \( c \). We know that with probability \((1 - \delta)\) one of the counters will be \( \epsilon n \) close to \( n \). Why?
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**Idea:** Repeat independently $c \log(1/\delta)$ times for some constant $c$. We know that with probability $(1 - \delta)$ one of the counters will be $\epsilon n$ close to $n$. Why? Which one should we pick?
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We know that with probability \( (1 - \delta) \) one of the counters will be \( \epsilon n \) close to \( n \). Why? Which one should we pick?

**Algorithm:** Output median of \( Z^{(1)}, Z^{(2)}, \ldots, Z^{(\ell)} \).
Let $Z'$ be median of the $\ell = c \log(1/\delta)$ independent estimators.

**Lemma**

$$\Pr[|Z' - n| \geq \epsilon n] \leq \delta.$$
Error reduction via median trick

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**Lemma**

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Let $A_i$ be event that estimate $Z^{(i)}$ is *bad*: that is, $|Z^{(i)} - n| > \epsilon n$. $\Pr[A_i] < 1/4$. Hence expected number of bad estimates is $\ell/4$. 


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- For median estimate to be bad, more than half of $A_i$’s have to be bad.
Let $Z'$ be median of the $\ell = c \log(1/\delta)$ independent estimators.

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- Using Chernoff bounds: probability of bad median is at most $2^{-c'\ell}$ for some constant $c'$. 

Summarizing

*Using variance reduction and median trick:* with \( O\left(\frac{1}{\epsilon^2} \log(1/\delta) \log \log n \right) \) bits one can maintain a \((1 - \epsilon)\)-factor estimate of the number of events with probability \((1 - \delta)\). This is a *generic* scheme that we will repeatedly use.

For counter one can do (much) better by changing algorithm and better analysis. See homework and references in notes.