F_2 Estimation and Intro to Sketching

Lecture 08

February 07, 2019

Part I

F_2 Estimation

Estimating F_2

- Stream consists of $e_1, e_2, \ldots, e_m$ where each e_i is an integer in $[n]$. We know n in advance (or an upper bound)
- Given a stream let f_i denote the frequency of i or number of times i is seen in the stream
- Consider vector $\mathbf{f} = (f_1, f_2, \ldots, f_n)$

Question: Estimate $F_2 = \sum_{i=1}^{m} f_i^2$ in small space.

Using generic AMS sampling scheme we can do this in $O(\sqrt{n} \log n)$ space. Can we do it better?

AMS Scheme for F_2

AMS-F_2-Estimate:

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Let h : [n] → {−1, 1} be chosen from
    a 4-wise independent hash family H.
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While (stream is not empty) do
    a_j is current item
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$$Z^2 = \sum_i f_i^2 Y_i^2 + 2 \sum_{i \neq j} f_i f_j Y_i Y_j$$

and hence

$$\mathbf{E}\left[Z^2\right] = \sum_i f_i^2 = F_2.$$

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$$\begin{aligned} E[Z^4] &= \sum_{i \in [n]} \sum_{j \in [n]} \sum_{k \in [n]} \sum_{\ell \in [n]} f_i f_j f_k f_\ell E[Y_i Y_j Y_k Y_\ell] \\ &= \sum_{i \in [n]} f_i^4 + 6 \sum_{i=1}^{n} \sum_{j=i+1}^{n} f_i^2 f_j^2. \end{aligned}$$

Variance

$$
\begin{aligned}
Var(Z^2) &= \mathbf{E}\left[Z^4\right] - (\mathbf{E}\left[Z^2\right])^2 \\
&= F_4 - F_2^2 + 6\sum_{i=1}^{n}\sum_{j=i+1}^{n} f_i^2 f_j^2 \\
&= F_4 - (F_4 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} f_i^2 f_j^2) + 6\sum_{i=1}^{n}\sum_{j=i+1}^{n} f_i^2 f_j^2 \\
&= 4\sum_{i=1}^{n}\sum_{j=i+1}^{n} f_i^2 f_j^2 \\
&\leq 2F_2^2.
\end{aligned}
$$

Averaging and median trick again

Output is Z^2: and $\mathbf{E}\left[Z^2\right] = F_2$ and $Var(Z^4) \leq 2F_2^2$

- Reduce variance by averaging $8/\epsilon^2$ independent estimates. Let Y be the averaged estimator.
- Apply Chebyshev to average estimator. $\Pr[|Y - F_2| \geq \epsilon F_2] \leq 1/4$.
- Reduce error probability to δ by independently doing $O(\log(1/\delta))$ estimators above.
- Total space $O(\log(1/\delta)\frac{1}{\epsilon^2}\log n)$

Geometric Interpretation

Observation: The estimation algorithm works even when f_i's can be negative. What does this mean?

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Richer model:

- Want to estimate a function of a vector $x \in \mathbb{R}^n$ which is initially assume to be the all $\mathbf{0}$'s vector. (previously we were thinking of the frequency vector f)
- Each element e_j of a stream is a tuple (i_j, Δ_j) where $i_j \in [n]$ and $\Delta_i \in \mathbb{R}$ is a real-value: this updates x_{i_j} to $x_{i_j} + \Delta_j$. (Δ_j can be positive or negative)

Algorithm revisited

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Claim: Output estimates $||x||_2^2$ where x is the vector at end of stream of updates.

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And as before one can show that $Var(Z^2) \leq 2(\mathbf{E}\left[Z^2\right])^2$.

Introduction to (Linear) Sketching

A *sketch* of a stream σ is a summary data structure $C(\sigma)$ (ideally of small space) such that the sketch of the composition $\sigma_1 \cdot \sigma_2$ of two streams σ_1 and σ_1 can be computed from $C(\sigma_1)$ and $C(\sigma_2)$. The output of the algorithm is some function of the sketch.

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Is the sketch for F_2 estimation a linear sketch?

F_2 Estimation as Linear Sketching

Recall that we take average of independent estimators and take median to reduce error. Can we view all this as a sketch?

```
AMS-ℓ2-Sketch:
    ℓ = c log(1/δ)/ε²
    Let M be a ℓ × n matrix with entries in {−1, 1} s.t
        (i) rows are independent and
        (ii) in each row entries are 4-wise independent
    z is a ℓ × 1 vector initialized to 0
    While (stream is not empty) do
        a_j = (i_j, Δ_j) is current update
        z ← z + Δ_j M e_{i_j}
    endWhile
    Output vector z as sketch.
```

M is compactly represented via ℓ hash functions, one per row, independently chosen from 4-wise independent hash familty.

An Application to Join Size Estimation

In Databases an important operation is the "join" operation

- A relation/table r of arity k consists of tuples of size k where each tuple element is from some given type. Example: (netid, uin, last name, first name, dob, address) in a student data base
- Given two relations r and s and a common attribute a one often needs to compute their join $r \bowtie s$ over some common attribute that they share
- $r \bowtie s$ can have size quadratic in size of r and s

Question: Estimate size of $r \bowtie s$ without computing it explicitly. Very useful in database query optimization.

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Estimating $r \bowtie r$ over an attribute a is same as F_2 estimation.
Why?

Sketching: a shift in perspective

- Sketching ideas have many powerful applications in theory and practice
- In particular linear sketches are powerful. Allows one to handle negative entries and deletions. Surprisingly linear sketches are feasible in several settings.
- Connected to dimension reduction (JL Lemma), subspace embeddings and other important topics