Exercise 1: Balls and bins. Consider the standard balls and bins process. A collection of $m$ identical balls are thrown into $n$ bins. Each ball is thrown independently into a bin chosen uniformly at random.

(a) What is the (precise) probability that a particular bin $i$ contains exactly $k$ balls at the end of the experiment.

(b) Let $X$ be the number of bins that contain exactly $k$ balls. What is the expected value of $X$?

(c) What is the variance of $X$?

Exercise 2: Randomized max cut. In the max cut problem, the input is a graph $G = (V, E)$ with $m = |E|$ edges and $n = |V|$ vertices, and the goal is to partition $V$ into two sets $(A, B)$ (where $B = V \setminus A$) maximizing the number of edges $\{e = (u, v) \in E : u \in A, v \in B\}$ with endpoints in different sets. (Such an edge is said to be cut by the partition $(A, B)$). This problem is known to be NP-Hard, but we will show that it is very easy to get a constant factor approximation.

(a) Consider the following randomized algorithm.

```
random-partition(G = (V, E))

1. A, B ← ∅
2. for each v ∈ V
   A. with probability 1/2
      i. A ← A ∪ {v}
   B. else B ← B ∪ {v}
3. return (A, B)
```

random-partition randomly partitions the vertices by assigning each vertex to $A$ or $B$ independently with equal probability. Show that this algorithm cuts $m/2$ edges in expectation.
(b) Let \( k \in \mathbb{N} \). In the \textbf{max k-cut} problem, we want to partition \( V \) into \( k \) sets \((A_1, \ldots, A_k)\) maximizing the number of edges with endpoints in different parts. Consider the following randomized algorithm.

\[
\text{random-}k\text{-partition}(G = (V, E))
\]

1. \( A_1, \ldots, A_k \leftarrow \emptyset \)
   
2. for each \( v \in V \)
   
   // \([k] = \{1, \ldots, k\}\)
   
   A. sample \( i \in [k] \) uniformly at random
   
   B. \( A_i \leftarrow A_i \cup \{v\} \)

3. return \((A_1, \ldots, A_k)\)

\text{random-}k\text{-partition} randomly partitions the vertices into \( k \) sets analogously to \text{random-partition}. Show that this algorithm cuts \((1 - 1/k)m\) edges in expectation.

**Exercise 3: Coupon Collectors.** In the coupon collectors problem, there are \( n \) coupons, and each round we are given one of the coupons uniformly at random. Coupons can repeat. We want to collect all \( n \) coupons, and in particular, we want to analyze the expected number of rounds before collecting all \( n \) coupons.

1. Suppose a coin flips heads with probability \( p \). Show that the expected number of coin tosses until flipping heads is \( 1/p \).

2. For \( i \in [n] \), show that the expected number of iterations between collecting the \((i − 1)\)th coupon and the \(i\)th coupon is \( \frac{n}{n + 1 − i} \).

3. Show that the expected number of iterations until collecting all \( n \) coupons is \( nH_n \), where \( H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \) is the \textbf{n}th \textbf{harmonic number} (and approximately \( \ln(n) \)).