Topics in Streaming

Lecture 18 and 19
October 27 and 29, 2020
Topics in Streaming

- $F_p$ estimation for $p \in (0, 2]$ via $p$-stable distributions and pseudorandom generators
- Priority Sampling
- Precision Sampling and Applications to $\ell_2$ sampling in streams
- $\ell_0$ Sampling
Part I

$F_p \text{ Estimation}$
For estimation and JL and Euclidean LSH we used important “stability” property of the Normal distribution.

**Lemma**

Let $Y_1, Y_2, \ldots, Y_d$ be independent random variables with distribution $\mathcal{N}(0, 1)$. $Z = \sum_i x_i Y_i$ has distribution $\|x\|_2 \mathcal{N}(0, 1)$.

Standard Gaussian is 2-stable.
A real-valued distribution $\mathcal{D}$ is $p$-stable if $Z = \sum_{i=1}^{n} x_i Y_i$ has distribution $\|x\|_p \mathcal{D}$ when the $Y_i$ are independent and each of them is distributed as $\mathcal{D}$.

$$\sum_{i=1}^{n} x_i Y_i \sim \|x\|_p Z \sim \mathcal{D}$$
**Definition**

A real-valued distribution $\mathcal{D}$ is $p$-stable if $Z = \sum_{i=1}^{n} x_i Y_i$ has distribution $\|x\|_p \mathcal{D}$ when the $Y_i$ are independent and each of them is distributed as $\mathcal{D}$.

**Question:** Do $p$-stable distributions exist for $p \neq 2$?
$p$-stable distributions

**Fact:** $p$-stable distributions exist for all $p \in (0, 2]$ and do not exist for $p > 2$.

$p = 1$ is the Cauchy distribution which is the distribution of the ratio of two independent Gaussian random variables. Has a closed form density function $\frac{1}{\pi(1+x^2)}$. Mean and variance are *not* finite.
**$p$-stable distributions**

**Fact:** $p$-stable distributions exist for all $p \in (0, 2]$ and do not exist for $p > 2$.

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For general $p$ no closed form formula for density but can sample from the distribution.
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Streaming, sketching, LSH ideas for $\ell_2$ generalize to $\ell_p$ for $p \in (0, 2]$ via $p$-stable distributions and additional technical work.
Sampling from $p$-stable distribution

For $p \in (0, 2]$ let $D_p$ denote $p$-stable distribution. Sampling from $D_p$ via Chambers-Mallows-Stuck method

- Sample $\theta$ uniformly from $[-\pi/2, \pi/2]$.
- Sample $r$ uniformly from $[0, 1]$.
- Output

$$\frac{\sin(p\theta)}{(\cos \theta)^{1/p}} \left( \frac{\cos((1 - p)\theta)}{\ln(1/r)} \right)^{(1-p)/p}.$$

$p$-stable distributions need not have finite mean/variance. Hence we need to work with median of distribution.

**Definition**

The median of a distribution $D$ is $\theta$ if for $Y \sim D$,

$$\Pr[Y \leq \mu] = 1/2.$$ If $\phi(x)$ is the probability density function of $D$ then we have $\int_{-\infty}^{\mu} \phi(x) \, dx = 1/2$. 
$X \sim N(0, 1)$
**F_p** estimation via **p**-stable distribution

For \( p \in (0, 2] \) due to [Indyk]

**F_p**-Estimate:

\[
 k \leftarrow \Theta\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right)
\]

Let \( M \) be a \( k \times n \) matrix where each \( M_{ij} \sim D_p \)

\( y \leftarrow Mx \)

Output \( Y \leftarrow \frac{\text{median}(|y_1|, |y_2|, \ldots, |y_k|)}{\text{median}(|D_p|)} \)

\[
\begin{pmatrix}
  \pm 1 \\
  \vdots \\
  \pm 1
\end{pmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
\]

\( \sim \|x\|p D_p \)

\( i, \Delta i \)

\( x_i \leftarrow x_i + \Delta i \)

\( \|x\|p \)

\( \mathbb{E} \left[ \gamma_1, \gamma_2, \ldots, \gamma_n \right] \leq \delta^{-1/3} \)
\[ M \hat{x} = \hat{y} \]

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_k
\end{bmatrix}
\]

\[ y_i \approx \|x\|_p \delta_p \]

\[ E[y_i] = 0 \]

\[ \|y_i\|^p \]

\[
\frac{\text{median}(y_1, y_2, \ldots, y_k)}{\text{median}(1, \delta_p)}
\]

\[ \approx \|x\|_p. \]
$F_p$ estimation via $p$-stable distribution

For $p \in (0, 2]$ due to [Indyk]

$$F_p\text{-Estimate:}$$

$$k \leftarrow \Theta\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right)$$

Let $M$ be a $k \times n$ matrix where each $M_{ij} \sim \mathcal{D}_p$

$y \leftarrow Mx$

Output $Y \leftarrow \frac{\text{median}(|y_1|, |y_2|, \ldots, |y_k|)}{\text{median}(|\mathcal{D}_p|)}$

- Each $y_j$ is distributed according to $\|x\|_p \mathcal{D}_p$
- Cannot take average of $|y_j|^p$ values since mean of distribution is not finite
- Take median of absolute values for $k$ independent copies and normalize by median of distribution
Concentration Lemma

**Lemma**

Let $\epsilon > 0$ and let $D$ be a distribution with density function $\phi$ and a unique median $\mu > 0$. Suppose $\phi$ is absolutely continuous on $[(1 - \epsilon)\mu, (1 + \epsilon)\mu]$ and let $\alpha = \min\{\phi(x) \mid x \in [(1 - \epsilon)\mu, (1 + \epsilon)\mu]\}$. Let $Y = \text{median}(Y_1, Y_2, \ldots, Y_k)$ where $Y_1, \ldots, Y_k$ are independent samples from the distribution $D$. Then

$$
\Pr[|Y - \mu| \geq \epsilon \mu] \leq 2e^{-\frac{2}{3}\epsilon^2 \mu^2 \alpha^2 k}.
$$

See notes for proof idea.
Pseudorandom generator for $F_p$ Estimation

For $F_p$ estimation we need $M_{i,j}$ to be independent randomly distributed according to $D_p$. Can use sampling from distribution even though it is not explicit.

How do we store $M$ in small space?

Recall that for $F_2$ estimation and sketching we used matrix $M$ where each row of $M$ had 4-wise independent random variables. Needed separate proof to argue correctness.

Is there an equivalent limited independence hashing based algorithm for $F_p$ estimation?
Pseudorandom generator for $F_p$ Estimation

For $F_p$ estimation we need $M_{i,j}$ to be independent randomly distributed according to $D_p$. Can use sampling from distribution even though it is not explicit.

How do we store $M$ in small space?

Recall that for $F_2$ estimation and sketching we used matrix $M$ where each row of $M$ had 4-wise independent random variables. Needed separate proof to argue correctness.

Is there an equivalent limited independence hashing based algorithm for $F_p$ estimation? No but can use a powerful pseudorandomness tool from TCS.
Pseudorandom generator

- \( P \) class of decision problems decided in poly time.
- \( RP \) class of decision problems decided in randomized poly time with one-sided error
- \( BPP \) class of decision problems decided in randomized poly time with two-sided error allowed
Pseudorandom generator

- $P$ class of decision problems decided in poly time.
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**Big Open Problem:** Is $BPP = P$? Equivalently can every randomized polynomial time algorithm be derandomized with only polynomial-factor slow down?
Pseudorandom generator

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- \( BPP \) class of decision problems decided in randomized poly time with two-sided error allowed

Big Open Problem: Is \( BPP = P \)? Equivalently can every randomized polynomial time algorithm be derandomized with only polynomial-factor slow down?

Equivalently: Is there a pseudo-random generator that fools every poly-sized algorithm?
Nisan’s pseudorandom generator

Nisan constructed explicit pseudo-random generator that fools space-bounded algorithms.

**Theorem**

Let $A$ be an algorithm that uses space at most $S(n)$ on an input of length $n$. Then there is a pseudo-random generator $G$ that fools $A$ and has seed length $\ell = O(S(n) \log n)$ and which is computable in $O(\ell)$ space and $\text{poly}(\ell)$ time.

**Corollary**

For $S(n) = O(\log^c n)$ the generator uses space $S(n) = O(\log^{c+1} n)$ and can generate any of the desired random pseudo-random bits for algorithm in $\text{poly}(\log n)$ time.
Applying Nisan’s generator as a hammer

At a high-level if a streaming algorithm uses small space (polylogarithmic in input size) assuming access to *true* random bits then one can use Nisan’s generator to reduce space.

- Nisan’s generator requires small random seed. Store it.
- Generate required (pseudo)random bits “on the fly”. Note that Nisan’s generator itself runs in small space so total space is small.

Note that algorithm still uses random bits!
Applying Nisan’s generator as a hammer

At a high-level if a streaming algorithm uses small space (polylogarithmic in input size) assuming access to true random bits then one can use Nisan’s generator to reduce space.

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With additional discretization tricks one can convert Indyk’s $F_p$ estimation algorithm via Nisan’s generator into a true small space algorithm.

[Kane-Nelson-Woodruff] show how to use limited independence hashing for $F_p$ estimation instead of above hammer.
Original

random bits

input

space

NE

KLONDIKE

is

"a"

New Modifier

true

pseudo random

A

A

space
Sampling for data reduction

- $X$ set of $n$ points in the plane $a_1, a_2, \ldots, a_n$.
- Want to answer queries of the form: given some shape $C$ (say circles), how many points inside $C$?
- standard data structures or brute force linear search say

$$\frac{n}{\frac{n}{k}} \times \frac{n}{1c}$$
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**Question:** Suppose \( n \) is too large and we can only store \( k \) points for some \( k < n \).

**Sampling approach:**
- \( S \) sample of size \( k \) (with replacement). Store only \( S \)
- Given query \( C \), compute \( |C \cap S| \). What should we report as an estimate for \( |C \cap X| \)?
Sampling for data reduction

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**Sampling approach:**

- $S$ sample of size $k$ (with replacement). Store only $S$
- Given query $C$, compute $|C \cap S|$. What should we report as an estimate for $|C \cap X|$? $\frac{n}{k}|C \cap S|$ which is an unbiased estimator
Weighted case

- $X$ set of $n$ points in the plane $a_1, a_2, \ldots, a_n$. Each point $a_i$ has a non-negative weight $w_i$.
- Want to answer queries of the form: given some shape $C$ (say circles), what is weight of point inside $C$?

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Weighted case

- $X$ set of $n$ points in the plane $a_1, a_2, \ldots, a_n$. Each point $a_i$ has a non-negative weight $w_i$
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**Question:** Suppose $n$ is too large and we can only store $k$ points for some $k < n$.

**Sampling approach?**

- Easy to see that uniform sampling is not ideal
- Sample in proportion to weight? Say $a_i$ sampled with $p_i = w_i/W$ where $W = \sum_i w_i$.
- What do we set the weight of the sampled points to? Can we control sample size? What is the variance?
Importance Sampling

- Decide sampling probabilities $p_1, p_2, \ldots, p_n$
- Choose $a_i$ independently with probability $p_i$ and if $i$ is chosen set $\hat{w}_i = w_i / p_i$. If $i$ is not chosen we implicitly set $\hat{w}_i = 0$.

$$E[\hat{w}_i] = p_i \cdot \frac{w_i}{p_i} = w_i$$
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- For any $i$, $E[\hat{w}_i] = w_i$. 

$$\sum_{i=1}^{n} \frac{n}{p_i} \rightarrow$$
Decide sampling probabilities $p_1, p_2, \ldots, p_n$

Choose $a_i$ independently with probability $p_i$ and if $i$ is chosen set $\hat{w}_i = w_i/p_i$. If $i$ is not chosen we implicitly set $\hat{w}_i = 0$.

For any $i$, $E[\hat{w}_i] = w_i$. Hence for any $C$, $E[\hat{w}(C \cap S)] = E[w(C \cap S)]$. 

Question: How should we choose $p_i$'s?

Choose to reduce variance for queries of interest (depends on queries). Expected number of chosen points is $P_i p_i$ and hence choose $p_i$'s to roughly meet the memory bound. If we have memory of size $k$ then can scale $p_i$ values (sampling rate) to achieve this.
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Importance Sampling in Streaming Setting

Setting:
- Points $a_1, \ldots, a_n$ with weights arriving in stream
- Have a memory size of $k$
- Want to maintain a $k$-sample (to utilize memory as well as possible) such that we can estimate $w(C \cap X)$ accurately
- Stream length unknown! How can we adjust sampling rate?

Given every $C$, reclaim $w_1, w_2, \ldots, w_n$
Priority Sampling

[Duffield, Lund, Thorup]

- Queries are arbitrary subset sums so no structure there to exploit
- Focus on streaming aspect and using memory

Scheme:

1. For each $i \in \{1, \ldots, n\}$ set priority $q_i = w_i / u_i$ where $u_i$ is chosen uniformly (and independently from other items) at random from $[0, 1]$.

2. $S$ is the set of items with the $k$ highest priorities.

3. $\tau$ is the $(k+1)$st highest priority. If $k = n$ we set $\tau = 0$.

4. If $i \in S$, set $\hat{w}_i = \max\{w_i, \tau\}$, else set $\hat{w}_i = 0$.

Claim: Can maintain $S$, $\tau$ in streaming setting
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4. If $i \in S$, set $\hat{w}_i = \max\{w_i, \tau\}$, else set $\hat{w}_i = 0$. 
\[
\begin{align*}
a_1, a_2, a_3, \ldots, a_n \\
u_1, u_2, \ldots, u_n \quad \text{indep} \\
0.1, 0.5, 0.4, 0.6 \\
\frac{1}{0.1}, \frac{1}{0.5}, \frac{1}{0.4}, \frac{1}{0.6} \quad k = 2
\end{align*}
\]

\[
S = \{a_1, a_{10}\} \quad \tau = \frac{1}{0.5}
\]

\[
\hat{\omega}_1 = \max \left\{ \omega_1, \tau^3 \right\} \quad \frac{1}{k+2}
\]

\[
\tau \approx ?
\]

\[
\text{Theoretically random #}
\]

\[
\text{n (0,1)} \text{ random #s}
\]

Want K+1st smallest value?

\[
\frac{k+2}{n} \quad \text{or } \tau = \frac{n}{k+2}
\]
Priority Sampling

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1. For each $i \in [n]$ set priority $q_i = w_i / u_i$ where $u_i$ is chosen uniformly (and independently from other items) at random from $[0, 1]$.
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3. $\tau$ is the $(k + 1)$'st highest priority. If $k \geq n$ we set $\tau = 0$.
4. If $i \in S$, set $\hat{w}_i = \max\{w_i, \tau\}$, else set $\hat{w}_i = 0$.

Claim: Can maintain $S, \tau$ in streaming setting
\[ a_1, a_2, \ldots, a_n \]
\[ \omega = 10, 5, 3, 2, 11 \]
\[ u = 0.3, 0.2, 0.5, 0.6 \]

\[ q_i = \frac{\omega_i}{u_i} \]

But and later k apply & elements

\[ \tau = k+1 \] highest priority

\[ \bar{\omega}_i = \max \{ \omega_i, \tau \} \]
Priority Sampling

Intuition: from uniform weight case

- Suppose $w_i = 1$ for all $i$. Then sampling $k$ without repetition can be done via adaptation of reservoir sampling.
- A different approach: pick a uniformly random $r_i \in [0, 1]$ for each $i$. And pick top $k$ in terms of $r_i$ values (simulates random permutation) but can be done in streaming fashion. Many other distributions would work too and picking top $k$ according to $1/r_i$ works too.
- Why $1/r_i$? What is the expected value of $\tau$?
Lemma

\[ E[\hat{w}_i] = w_i. \]
Priority Sampling: Properties

**Lemma**

\[ E[\hat{w}_i] = w_i. \]

**Lemma**

\[ \text{Var}[\hat{w}_i] = E[\hat{v}_i] \quad \text{where} \quad \hat{v}_i = \begin{cases} \tau \max\{0, \tau - w_i\} & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \]

Useful: storing \( \tau \) and \( w_i \) gives \( \text{Var}[\hat{w}_i] \).
Priority Sampling: Properties

Lemma
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Lemma
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Useful: storing \( \tau \) and \( w_i \) gives \( \text{Var}[\hat{w}_i] \).

Lemma
If \( k \geq 2 \) for any \( i \neq j \), \( \mathbb{E}[\hat{w}_i \hat{w}_j] = w_i w_j. \)

Lemma
Fix any set \( C \subseteq [n] \). \( \mathbb{E}\left[ \prod_{i \in C} \hat{w}_i \right] = \prod_{i \in C} w_i \) if \( |C| \leq k \) and is 0 if \( |C| > k. \)
Variance of subset sum

Lemma

If \( k \geq 2 \) for any \( i \neq j \), \( \mathbb{E}[\hat{w}_i \hat{w}_j] = w_i w_j \).

Consequence:

- Fix \( C \). Unbiased estimator of \( w(C \cap X) \) is \( \hat{w}(C \cap S) \).
- Can we know the variance of the estimate to know if we are doing ok?
- \( \text{Var}[\hat{w}(C \cap S)] = \sum_{i \in C \cap S} \text{Var}[\hat{w}_i] = \sum_{i \in C \cap S} \mathbb{E}[\hat{v}_i] \). Hence, storing \( \tau \) and \( \hat{w}_i \) values suffices to estimate the variance of the estimate.
Lemma

$E[\hat{w}_i] = w_i$.  

Fix $i$. Let $A(\tau_0)$ be the event that the $k$'th highest priority among items $j \neq i$ is $\tau_0$. Note that $u_i$ is independent of $\tau_0$. Hence $i \in S$ if $q_i = w_i/u_i$ and if $i \in S$ then $\hat{w}_i = \max\{w_i, \tau_0\}$, otherwise $\hat{w}_i = 0$. To evaluate $Pr[i \in S | A(\tau_0)]$ we consider two cases.

Case 1: $w_i > \tau_0$. Here we have $Pr[i \in S | A(\tau_0)] = 1$ and $\hat{w}_i = w_i$.

Case 2: $w_i < \tau_0$. Then $Pr[i \in S | A(\tau_0)] = w_i/\tau_0$ and $\hat{w}_i = \tau_0$.

In both cases we see that $E[\hat{w}_i] = w_i$.  

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Priority Sampling: Properties

Lemma

$E[\hat{w}_i] = w_i$.

Fix $i$. Let $A(\tau')$ be the event that the $k$'th highest priority among items $j \neq i$ is $\tau'$. Note that $u_i$ is independent of $\tau'$. Hence $i \in S$ if $q_i = w_i/u_i \geq \tau'$ and if $i \in S$ then $\hat{w}_i = \max\{w_i, \tau'\}$, otherwise $\hat{w}_i = 0$. To evaluate $Pr[i \in S \mid A(\tau')]$ we consider two cases.

Case 1: $w_i \geq \tau'$. Here we have $Pr[i \in S \mid A(\tau')] = 1$ and $\hat{w}_i = w_i$.

Case 2: $w_i < \tau'$. Then $Pr[i \in S \mid A(\tau')] = \frac{w_i}{\tau'}$ and $\hat{w}_i = \tau'$.

In both cases we see that $E[\hat{w}_i] = w_i$. 
Variance

Lemma

\[ \text{Var}[\hat{w}_i] = \mathbb{E}[\hat{v}_i] \text{ where } \hat{v}_i = \begin{cases} \tau \max\{0, \tau - w_i\} & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \]
Variance

**Lemma**

\[ \text{Var}[\hat{w}_i] = E[\hat{v}_i] \quad \text{where} \quad \hat{v}_i = \begin{cases} \tau \max\{0, \tau - w_i\} & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \]

Fix \( i \). We define \( A(\tau') \) to be the event that \( \tau' \) is the \( k' \)th highest priority among elements \( j \neq i \).

Show that

\[ E[\hat{v}_i \mid A(\tau')] = E[\hat{w}_i^2 \mid A(\tau')] - w_i^2. \]

Since \( u_i \) is independent of \( \tau' \) we can remove conditioning
Variance

\[ E[\hat{v}_i \mid A(\tau')] = E[\hat{w}_i^2 \mid A(\tau')] - w_i^2. \]

\[ E[\hat{v}_i \mid A(\tau')] = \Pr[i \in S \mid A(\tau')] \times E[\hat{v}_i \mid i \in S \land A(\tau')] \]
\[ = \min\{1, \frac{w_i}{\tau'}\} \times \tau' \max\{0, \tau' - w_i\} \]
\[ = \max\{0, w_i \tau' - w_i^2\}. \]
Variance

\[ E[\hat{v}_i \mid A(\tau')] = E[\hat{w}_i^2 \mid A(\tau')] - w_i^2. \]

\[ E[\hat{v}_i \mid A(\tau')] = \text{Pr}[i \in S \mid A(\tau')] \times E[\hat{v}_i \mid i \in S \wedge A(\tau')] \]
\[ = \min\{1, \frac{w_i}{\tau'}\} \times \tau' \max\{0, \tau' - w_i\} \]
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\[ = \min\{1, \frac{w_i}{\tau'}\} \times (\max\{w_i, \tau'\})^2 \]
\[ = \max\{w_i^2, w_i \tau'\}. \]
Lemma

If \( k \geq 2 \) for any \( i \neq j \), \( \mathbb{E}[\hat{w}_i \hat{w}_j] = w_i w_j \).

More generally

Lemma

Fix any set \( C \subset [n] \). \( \mathbb{E}[\prod_{i \in C} \hat{w}_i] = \prod_{i \in C} w_i \) if \( |C| \leq k \) and is \( 0 \) if \( |C| > k \).
Variance of subset sum

**Lemma**

If $k \geq 2$ for any $i \neq j$, $E[\hat{w}_i \hat{w}_j] = w_i w_j$.

More generally

**Lemma**

Fix any set $C \subseteq [n]$. $E[\prod_{i \in C} \hat{w}_i] = \prod_{i \in C} w_i$ if $|C| \leq k$ and is 0 if $|C| > k$.

Requires a proof by induction. See notes
Variance of subset sum

Lemma

If \( k \geq 2 \) for any \( i \neq j \), \( \mathbb{E}[\hat{w}_i\hat{w}_j] = w_iw_j \).

More generally

Lemma

Fix any set \( C \subset [n] \). \( \mathbb{E}[\prod_{i \in C} \hat{w}_i] = \prod_{i \in C} w_i \) if \( |C| \leq k \) and is 0 if \( |C| > k \).

Requires a proof by induction. See notes

Why is this interesting/non-obvious? In vanilla importance sampling the variables \( \hat{w}_i \) are independent. However, here the variables are correlated because we choose exactly \( k \). Nevertheless, they exhibit properties similar to independence.
Application of $\ell_2$ sampling to $F_p$ estimation

For $p > 2$ AMS-Sampling gives algorithm to estimate $F_p$ using $\tilde{O}(n^{1-1/p})$ space. Optimal space is $\tilde{O}(n^{1-2/p})$.

- Use $\ell_2$ sampling algorithm to generate $(i, |\tilde{x}_i|)$
- Estimate $\|x\|_2$
- Output $T = \|x_2\|^2 |\tilde{x}_i|^{p-2}$ as estimate

To simplify analysis/notation assume sampling is exact.

$$E[T] = \|x\|_2^2 \sum_i \frac{x_i^2}{\|x\|_2^2} |x_i|^{p-2} = \sum_i |x_i|^p$$

$$\text{Var}[T] \leq \|x\|_2^4 \sum_i \frac{x_i^2}{\|x\|_2^2} x_i^{2(p-2)} \leq \|x\|_2^2 \sum_i x_i^{2p-2} \leq n^{1-2/p} (\sum_i |x_i|^p)^2.$$  

Now do average plus median.