Richer model:

- Want to estimate a function of a vector $\mathbf{x} \in \mathbb{R}^n$ which is initially assume to be the all 0’s vector.

- Each element $e_j$ of a stream is a tuple $(i_j, \Delta_j)$ where $i_j \in [n]$ and $\Delta_j \in \mathbb{R}$ is a real-value: this updates $x_{i_j}$ to $x_{i_j} + \Delta_j$. ($\Delta_j$ can be positive or negative)
Models

Richer model:

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- $\Delta_j > 0$: *cash register* model. Special case is $\Delta_j = 1$.
- $\Delta_j$ arbitrary: *turnstile* model
- $\Delta_j$ arbitrary but $x \geq 0$ at all times: *strict turnstile* model
- *Sliding window* model: interested only in the last $W$ items (window)
Frequent Items Problem

What is $F_k$ when $k = \infty$?
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Heavy Hitters Problem: Find all items $i$ such that $f_i > m/k$ for some fixed $k$.

Heavy hitters are very frequent items.
Majority element problem:

- Offline: given an array/list $A$ of $m$ integers, is there an element that occurs more than $m/2$ times in $A$?
- Streaming: is there an $i$ such that $f_i > m/2$?
Finding Majority Element

Streaming-Majority:

\[ c = 0, \ s \leftarrow \text{null} \]

While (stream is not empty) do

\[ \text{If (} e_j = s \text{) do} \]

\[ c \leftarrow c + 1 \]

\[ \text{ElseIf (} c = 0 \text{)} \]

\[ c = 1 \]

\[ s = e_j \]

\[ \text{Else} \]

\[ c \leftarrow c - 1 \]

endWhile

Output \( s, c \)

Claim:
If there is a majority element \( i \) then algorithm outputs \( s = i \) and \( c \geq f_i - m / 2 \).

Caveat: Algorithm may output incorrect element if no majority element. Can verify correctness in a second pass.
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Misra-Gries Algorithm

**Heavy Hitters Problem:** Find all items $i$ such that $f_i > m/k$.

**MisraGreis($k$):**

- $D$ is an empty associative array
- While (stream is not empty) do
  - $e_j$ is current item
  - If ($e_j$ is in $keys(D)$)
    - $D[e_j] \leftarrow D[e_j] + 1$
  - Else if ($|keys(A)| < k - 1$) then
    - $D[e_j] \leftarrow 1$
  - Else
    - for each $\ell \in keys(D)$ do
      - $D[\ell] \leftarrow D[\ell] - 1$
    - Remove elements from $D$ whose counter values are 0
- endWhile
- For each $i \in keys(D)$ set $\hat{f}_i = D[i]$
- For each $i \not\in keys(D)$ set $\hat{f}_i = 0$
Analysis

Space usage $O(k)$.

**Theorem**

For each $i \in [n]$: $f_i - \frac{m}{k+1} \leq \hat{f}_i \leq f_i$.

**Corollary**

Any item with $f_i > \frac{m}{k}$ is in $D$ at the end of the algorithm.

A second pass to verify can be used to verify correctness of elements in $D$. 
Proof of Correctness

**Theorem**

For each $i \in [n]$: $f_i - \frac{m}{k+1} \leq \hat{f}_i \leq f_i$.
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Easy to see: \( \hat{f}_i \leq f_i \). Why?
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Alternative view of algorithm:

- Maintains counts $C[i]$ for each $i$ (initialized to 0). Only $k$ are non-zero at any time.
- When new element $e_j$ comes
  - If $C[e_j] > 0$ then increment $C[e_j]$
  - Elself less then $k$ positive counters then set $C[e_j] = 1$
  - Else decrement all positive counters (exactly $k$ of them)
- Output $\hat{f}_i = C[i]$ for each $i$
Proof of Correctness

Want to show: \( f_i - \hat{f}_i \leq m/(k + 1) \):

Suppose we have \( \ell \) occurrences of \( k \) counters being decremented. Then \( \ell k + \ell \leq m \) which implies \( \ell \leq m/(k + 1) \).

Consider \( \alpha = (f_i - \hat{f}_i) \) as items are processed. Initially 0. How big can it get?

If \( e_j = i \) and \( C[i] \) is incremented \( \alpha \) stays the same.

If \( e_j = i \) and \( C[i] \) is not incremented then \( \alpha \) increases by one and \( k \) counters decremented — charge to \( \ell \).

If \( e_j \neq i \) and \( \alpha \) increases by 1 it is because \( C[i] \) is decremented — charge to \( \ell \).

Hence total number of times \( \alpha \) increases is at most \( \ell \).
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- Hence total number of times \( \alpha \) increases is at most \( \ell \).
Deterministic to Randomized Sketches

Cannot improve $O(k)$ space if one wants additive error of at most $m/k$. Nice to have a deterministic algorithm that is near-optimal

Why look for randomized solution?
- Obtain a sketch that allows for deletions
- Additional applications of sketch based solutions
- Will see Count-Min and Count sketches
Basic Hashing/Sampling Idea

**Heavy Hitters Problem:** Find all items $i$ such that $f_i > m/k$.

- Let $b_1, b_2, \ldots, b_k$ be the $k$ heavy hitters.
- Suppose we pick $h : [n] \rightarrow [ck]$ for some $c > 1$.
- $h$ spreads $b_1, \ldots, b_k$ among the buckets ($k$ balls into $ck$ bins).
- In ideal situation each bucket can be used to count a separate heavy hitter.