1. Given an undirected graph $G = (V, E)$ a matching M is a set of edges from E such that no two edges in M share an end point. A matching is perfect if $|M| = |V|/2$, that is, every vertex is matched by M. It turns out that finding maximum matchings and perfect matchings in bipartite graphs[1] is easier than in general graphs. The goal of this problem is to describe an *incorrect* reduction to point out that the proof of correctness of a reduction involves showing two directions. Here is the incorrect reduction. Given an arbitrary graph $G = (V, E)$ create a bipartite graph $G' = (V_1, V_2, E')$ where V_1 and V_2 are copies of V. Formally $V_1 = \{u^{(1)} \mid u \in V\}$ and $V_2 = \{u^{(2)} \mid u \in V\}$. For every edge $(u, v) \in E$ we add two edges $(u^{(1)}, v^{(2)})$ and $(v^{(1)}, u^{(2)})$ in E'.
   - Show that G has a matching of size k implies that G' has a matching of size at least $2k$.
   - Give an example where G' has a matching of size at least $2k$ but G does not have a matching of size k. *Hint:* Consider G to be the union of two disjoint triangles.

2. Self-reduction. We focus on decision problems even when the underlying problem we are interested in is an optimization problem. For most problems of interest we can in fact show that a polynomial-time algorithm for the decision problem also implies a polynomial-time algorithm for the corresponding optimization problem. To illustrate this consider the maximum independent set (MIS) problem.
   - Suppose you are given a algorithm that given a graph H and integer ℓ outputs whether H has an independent set of size at least ℓ. Using this algorithm as a *black box*, describe a polynomial time algorithm that given a graph G and integer k outputs an independent set of size k in G if it has one. Note that you can use the black box algorithm more than once. *Hint:* What happens if you remove a vertex v and the independent set size does not decrease? What if it does?
   - How would you efficiently find a maximum independent set in a given graph G using the black box?

3. A cycle C in a directed graph G is called a Hamiltonian cycle if it contains all the vertices of G. The Hamiltonian Cycle problem is the following: given G, does G contain a Hamiltonian cycle? The Longest Path problem is the following: given a directed graph G and integer k, is there a simple path of length k in G? Assuming that you have a black box algorithm for the Longest Path problem describe a polynomial-time algorithm for the Hamiltonian Cycle problem. *Prove* the correctness of your algorithm.

[1] A graph $G = (V, E)$ is bipartite if V can be partitioned into V_1 and V_2 such that all edges have one end point in V_1 and the other in V_2; that is, V_1 and V_2 are independent sets.