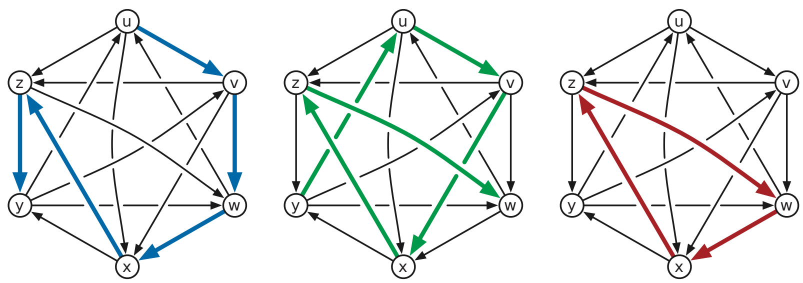
1. Suppose you have a rectangular bar of chocolate, which has been scored into an $n \times m$ grid of squares. Consider breaking the chocolate into squares in the following way. In each round you take one of the available pieces of chocolate and break it along one of the grid lines into two smaller rectangles. Thus, at all times, each piece of chocolate is an $a \times b$ rectangle for some positive integers $a$ and $b$; in particular, a $1 \times 1$ piece cannot be broken into smaller pieces. The process ends when all the pieces are individual squares. Prove by induction that, no matter the strategy, the number of rounds to break the chocolate is $n \times m - 1$.

2. Prove that every non-negative integer can be represented as the sum of distinct powers of 2. ("Write it in binary" is not a proof; it's just a restatement of what you have to prove.)

3. **[To think about later]** A ***tournament*** is a directed graph with exactly one directed edge between each pair of vertices. That is, for any vertices $v$ and $w$, a tournament contains either an edge $v \to w$ or an edge $w \to v$, but not both. A ***Hamiltonian path*** in a directed graph $G$ is a directed path that visits every vertex of $G$ exactly once.




A tournament with two Hamiltonian paths $u \to v \to w \to x \to z \to y$ and $y \to u \to v \to x \to z \to w$ and a directed triangle $w \to x \to z \to w$.

Prove that every tournament contains *at least* one Hamiltonian path.