CS 374: Algorithms & Models of Computation

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Graphs, Representation, Search, DFS

Lecture 8
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Why Graphs?

1. Graphs help model networks which are ubiquitous: transportation networks (rail, roads, airways), social networks (interpersonal relationships), information networks (web page links), and many problems that don’t even look like graph problems.

2. Fundamental objects in Computer Science, Optimization, Combinatorics

3. Many important and useful optimization problems are graph problems

4. Graph theory: elegant, fun and deep mathematics
Graph

Definition

An undirected (simple) graph \( G = (V, E) \) is a 2-tuple:

1. \( V \) is a set of vertices (also referred to as nodes/points)
2. \( E \) is a set of edges where each edge \( e \in E \) is a set of the form \( \{u, v\} \) with \( u, v \in V \) and \( u \neq v \).

Example

In figure, \( G = (V, E) \) where \( V = \{1, 2, 3, 4, 5, 6, 7, 8\} \) and \( E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{7, 8\}\} \).
Example: Modeling Problems as Search

State Space Search

Many search problems can be modeled as search on a graph. The trick is figuring out what the vertices and edges are.

Missionaries and Cannibals

- Three missionaries, three cannibals, one boat, one river
- Boat carries two people, must have at least one person
- Must all get across
- At no time can cannibals outnumber missionaries

How is this a graph search problem?
What are the vertices?
What are the edges?
Example: Missionaries and Cannibals Graph
Notation and Convention

Notation

An edge in an undirected graphs is an *unordered* pair of nodes and hence it is a set. Conventionally we use \((u, v)\) for \(\{u, v\}\) when it is clear from the context that the graph is undirected.

1. \(u\) and \(v\) are the end points of an edge \(\{u, v\}\)
2. Multi-graphs allow
   1. *loops* which are edges with the same node appearing as both end points
   2. *multi-edges*: different edges between same pairs of nodes
3. In this class we will assume that a graph is a simple graph unless explicitly stated otherwise.
### Adjacency Matrix

Represent $G = (V, E)$ with $n$ vertices and $m$ edges using a $n \times n$ adjacency matrix $A$ where:

2. Advantage: can check if $\{i, j\} \in E$ in $O(1)$ time.
3. Disadvantage: needs $\Omega(n^2)$ space even when $m \ll n^2$.
Adjacency Lists

Represent $G = (V, E)$ with $n$ vertices and $m$ edges using adjacency lists:

1. For each $u \in V$, $\text{Adj}(u) = \{v \mid \{u, v\} \in E\}$, that is neighbors of $u$. Sometimes $\text{Adj}(u)$ is the list of edges incident to $u$.

2. Advantage: space is $O(m + n)$

3. Disadvantage: cannot “easily” determine in $O(1)$ time whether $\{i, j\} \in E$
   - By sorting each list, one can achieve $O(\log n)$ time
   - By hashing “appropriately”, one can achieve $O(1)$ time

**Note:** In this class we will assume that by default, graphs are represented using plain vanilla (unsorted) adjacency lists.
A Concrete Representation

- Assume vertices are numbered arbitrarily as \( \{1, 2, \ldots, n\} \).
- Edges are numbered arbitrarily as \( \{1, 2, \ldots, m\} \).
- Edges stored in an array/list of size \( m \). \( E[j] \) is \( j \)'th edge with info on end points which are integers in range 1 to \( n \).
- Array \textbf{Adj} of size \( n \) for adjacency lists. \( \text{Adj}[i] \) points to adjacency list of vertex \( i \). \( \text{Adj}[i] \) is a list of edge indices in range 1 to \( m \).
A Concrete Representation

Array of edges E

\[ \begin{array}{c|c|c} \hline & e_j & \hline \end{array} \]

Information including end point indices

Array of adjacency lists

\[ \begin{array}{c|c} \hline \vdots & \hline v_i & \hline \vdots \end{array} \]

List of edges (indices) that are incident to \( v_i \)
A Concrete Representation: Advantages

- Edges are explicitly represented/numbered. Scanning/processing all edges easy to do.
- Representation easily supports multigraphs including self-loops.
- Explicit numbering of vertices and edges allows use of arrays: $O(1)$-time operations are easy to understand.
- Can also implement via pointer based lists for certain dynamic graph settings.
Connectivity

Given a graph $G = (V, E)$:

1. A path is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k - 1$. The length of the path is $k - 1$ (the number of edges in the path) and the path is from $v_1$ to $v_k$. Note: a single vertex $u$ is a path of length 0.

2. A cycle is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k - 1$ and $\{v_1, v_k\} \in E$. Single vertex not a cycle according to this definition. Caveat: Some times people use the term cycle to also allow vertices to be repeated; we will use the term tour.

3. A vertex $u$ is connected to $v$ if there is a path from $u$ to $v$.

4. The connected component of $u$, $\text{con}(u)$, is the set of all vertices connected to $u$. Is $u \in \text{con}(u)$?
Connectivity

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Connectivity contd

Define a relation \( C \) on \( V \times V \) as \( uCv \) if \( u \)
is connected to \( v \)

1. In undirected graphs, connectivity is
   a reflexive, symmetric, and transitive
   relation. Connected components are
   the equivalence classes.

2. Graph is connected if only one
   connected component.

In other words, every pair of vertices in
the graph are connected.
Connectivity Problems

Algorithmic Problems

1. Given graph $G$ and nodes $u$ and $v$, is $u$ connected to $v$?
2. Given $G$ and node $u$, find all nodes that are connected to $u$.
3. Find all connected components of $G$.

Can be accomplished in $O(m + n)$ time using BFS or DFS.
## Connectivity Problems

### Algorithmic Problems

1. Given graph $G$ and nodes $u$ and $v$, is $u$ connected to $v$?
2. Given $G$ and node $u$, find all nodes that are connected to $u$.
3. Find all connected components of $G$.

Can be accomplished in $O(m + n)$ time using **BFS** or **DFS**.
Basic Graph Search

Given $G = (V, E)$ and vertex $u \in V$:

**Explore**($u$):
- Initialize $S = \{u\}$
- while there is an edge $(x, y)$ with $x \in S$ and $y \notin S$ do
  - add $y$ to $S$
Basic Graph Search

Given $G = (V, E)$ and vertex $u \in V$:

**Explore**($u$):
- Initialize $S = \{u\}$
- while there is an edge $(x, y)$ with $x \in S$ and $y \not\in S$ do
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**Proposition**

**Explore**(u) terminates with $S = \text{con}(u)$. 
Basic Graph Search

Given \( G = (V, E) \) and vertex \( u \in V \):

**Explore**\((u)\):

Initialize \( S = \{u\} \)

while there is an edge \((x, y)\) with \( x \in S \) and \( y \not\in S \) do

add \( y \) to \( S \)

**Proposition**

**Explore**\((u) \) terminates with \( S = \text{con}(u) \).

Running time: depends on implementation

1. Naive: \( O(mn) \) with \( O(m) \) time for each scan.
2. Breadth First Search (**BFS**): use queue data structure
3. Depth First Search (**DFS**): use stack data structure
4. DFS/BFS run in \( O(m + n) \) time. Review CS 225 material!
Part II

DFS
Depth First Search

**DFS** is a very versatile graph exploration strategy. Hopcroft and Tarjan (Turing Award winners) demonstrated the power of **DFS** to understand graph structure. **DFS** can be used to obtain linear time \((O(m + n))\) algorithms for

1. Finding cut-edges and cut-vertices of undirected graphs
2. Finding strong connected components of directed graphs
3. Linear time algorithm for testing whether a graph is planar
DFS in Undirected Graphs

Recursive version.

\[
\text{DFS}(G) \\
\text{Mark all nodes as unvisited} \\
\text{while there is an unvisited node } u \ \text{do} \\
\text{DFS}(u)
\]

\[
\text{DFS}(u) \\
\text{Mark } u \text{ as visited} \\
\text{for each edge } (u,v) \text{ in Adj}(u) \ \text{do} \\
\text{if } v \text{ is not marked} \\
\text{DFS}(v)
\]

Implemented using a global array Mark for all recursive calls.
Example

```
Example

The set of connected components of a graph is the set \{\{u) | u \in V}\}

The connected components in the above graph are \{1, 2, 3, 4, 5, 6, 7, 8\} and \{9, 10\}.

A graph is said to be connected when it has exactly one connected component.
In other words, every pair of vertices in the graph are connected.
```
DFS(G)
Mark all nodes unvisited
Set $T$ to be empty
while $\exists$ unvisited node $u$ do
  DFS($u$)
Output $T$

DFS($u$)
Mark $u$ as visited
for $uv$ in $\text{Adj}(u)$ do
  if $v$ is not marked
    add $uv$ to $T$
  DFS($v$)
DFS Tree/Forest

**DFS(G)**
Mark all nodes unvisited
Set $T$ to be empty

while $\exists$ unvisited node $u$ do
    DFS($u$)

Output $T$

**DFS(u)**
Mark $u$ as visited

for $uv$ in Adj($u$) do
    if $v$ is not marked
        add $uv$ to $T$
    DFS($v$)

---

Edges classified into two types: $uv \in E$ is a

1. **tree edge:** belongs to $T$
2. **non-tree edge:** does not belong to $T$
Properties of DFS tree

Proposition

1. \( T \) is a forest
2. connected components of \( T \) are same as those of \( G \).
3. If \( uv \in E \) is a non-tree edge then, in \( T \), either:
   1. \( u \) is an ancestor of \( v \), or
   2. \( v \) is an ancestor of \( u \).

Question: Why are there no cross-edges?
DFS with Predecessors

Keep track of predecessors.

**DFS(G)**

```plaintext
for all u ∈ V(G) do
    Mark u as unvisited
    Set pred(u) to null
T is set to ∅
while ∃unvisited u do
    DFS(u)
Output T
```

**DFS(u)**

```plaintext
Mark u as visited
for each uv in Out(u) do
    if v is not marked then
        add edge uv to T
        set pred(v) to u
    DFS(v)
```
DFS with Visit Times

Keep track of when nodes are visited.

\[
\text{DFS}(G) \\
\text{for all } u \in V(G) \text{ do} \\
\quad \text{Mark } u \text{ as unvisited} \\
\quad T \text{ is set to } \emptyset \\
\quad \text{time} = 0 \\
\text{while } \exists \text{ unvisited } u \text{ do} \\
\quad \text{DFS}(u) \\
\quad \text{Output } T 
\]

\[
\text{DFS}(u) \\
\quad \text{Mark } u \text{ as visited} \\
\quad \text{pre}(u) = ++\text{time} \\
\quad \text{for each } uv \text{ in Out}(u) \text{ do} \\
\quad \quad \text{if } v \text{ is not marked then} \\
\quad \quad \quad \text{add edge } uv \text{ to } T \\
\quad \quad \text{DFS}(v) \\
\quad \text{post}(u) = ++\text{time} 
\]
The set of connected components of a graph is the set \( \{ \text{component} \mid u \in V \} \).

The connected components in the above graph are \( \{1, 2, 3, 4, 5, 6, 7, 8\} \) and \( \{9, 10\} \).

A graph is said to be connected when it has exactly one connected component. In other words, every pair of vertices in the graph are connected.
**Example**

The set of connected components of a graph is the set \( \{ u \mid u \in V \} \).

The connected components in the above graph are \( \{1, 2, 3, 4, 5\} \) and \( \{6\} \) and \( \{7, 8, 9, 10\} \).

A graph is said to be connected when it has exactly one connected component. In other words, every pair of vertices in the graph are connected.
pre and post numbers

Node $u$ is active in time interval $[\text{pre}(u), \text{post}(u)]$

**Proposition**

For any two nodes $u$ and $v$, the two intervals $[\text{pre}(u), \text{post}(u)]$ and $[\text{pre}(v), \text{post}(v)]$ are disjoint or one is contained in the other.
Node $u$ is **active** in time interval $[\text{pre}(u), \text{post}(u)]$

**Proposition**

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**Proof.**
Node $u$ is **active** in time interval $[\text{pre}(u), \text{post}(u)]$.

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**Proof.**

Assume without loss of generality that $\text{pre}(u) < \text{pre}(v)$. Then $v$ visited after $u$. 

*pre and post numbers*
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**Proof.**

- Assume without loss of generality that $\text{pre}(u) < \text{pre}(v)$. Then $v$ visited after $u$.
- If $\text{DFS}(v)$ invoked before $\text{DFS}(u)$ finished, $\text{post}(v) < \text{post}(u)$. 
**pre and post numbers**

Node $u$ is **active** in time interval $[\text{pre}(u), \text{post}(u)]$

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- If $\text{DFS}(v)$ invoked before $\text{DFS}(u)$ finished, $\text{post}(v) < \text{post}(u)$.
- If $\text{DFS}(v)$ invoked after $\text{DFS}(u)$ finished, $\text{pre}(v) > \text{post}(u)$. \(\square\)
Node \( u \) is active in time interval \([\text{pre}(u), \text{post}(u)]\)

**Proposition**

For any two nodes \( u \) and \( v \), the two intervals \([\text{pre}(u), \text{post}(u)]\) and \([\text{pre}(v), \text{post}(v)]\) are disjoint or one is contained in the other.

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- If \( \text{DFS}(v) \) invoked after \( \text{DFS}(u) \) finished, \( \text{pre}(v) > \text{post}(u) \).

\( \text{pre} \) and \( \text{post} \) numbers useful in several applications of \( \text{DFS} \)- soon!
Part III

Directed Graphs and Decomposition
Directed Graphs

**Definition**

A directed graph $G = (V, E)$ consists of

1. set of vertices/nodes $V$
2. a set of edges/arcs $E \subseteq V \times V$.

An edge is an *ordered* pair of vertices. $(u, v)$ different from $(v, u)$. 

![Diagram of directed graph](image)
Examples of Directed Graphs

In many situations relationship between vertices is asymmetric:

1. Road networks with one-way streets.
2. Web-link graph: vertices are web-pages and there is an edge from page $p$ to page $p'$ if $p$ has a link to $p'$. Web graphs used by Google with PageRank algorithm to rank pages.
3. Dependency graphs in variety of applications: link from $x$ to $y$ if $y$ depends on $x$. Make files for compiling programs.
4. Program Analysis: functions/procedures are vertices and there is an edge from $x$ to $y$ if $x$ calls $y$. 
Graph $G = (V, E)$ with $n$ vertices and $m$ edges:


2. **Adjacency Lists**: for each node $u$, $\text{Out}(u)$ (also referred to as $\text{Adj}(u)$) and $\text{In}(u)$ store out-going edges and in-coming edges from $u$.

Default representation is adjacency lists. Concrete representation discussed previously for undirected graphs easily extends to directed graphs.
Directed Connectivity

Given a graph $G = (V, E)$:

1. A **(directed) path** is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$. The length of the path is $k - 1$ and the path is from $v_1$ to $v_k$. By convention, a single node $u$ is a path of length 0.

2. A **cycle** is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$ and $(v_k, v_1) \in E$. By convention, a single node $u$ is not a cycle.
Directed Connectivity

Given a graph $G = (V, E)$:

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3. A vertex $u$ can reach $v$ if there is a path from $u$ to $v$. Alternatively $v$ can be reached from $u$. 

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Directed Connectivity

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3. A vertex $u$ can **reach** $v$ if there is a path from $u$ to $v$. Alternatively $v$ can be reached from $u$.

4. Let $rch(u)$ be the set of all vertices reachable from $u$. 
Asymmetricity: D can reach B but B cannot reach D
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Questions:
1. Is there a notion of connected components?
2. How do we understand connectivity in directed graphs?
Connectivity and Strong Connected Components

Definition

Given a directed graph $G$, $u$ is strongly connected to $v$ if $u$ can reach $v$ and $v$ can reach $u$. In other words $v \in rch(u)$ and $u \in rch(v)$. 

Proposition

$C$ is an equivalence relation, that is reflexive, symmetric and transitive.

Equivalence classes of $C$: strong connected components of $G$. They partition the vertices of $G$.

$SCC(u)$: strongly connected component containing $u$. 

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Connectivity and Strong Connected Components

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Define relation $C$ where $uCv$ if $u$ is (strongly) connected to $v$.
Connectivity and Strong Connected Components

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$SCC(u)$: strongly connected component containing $u$. 
Strongly Connected Components: Example

A directed graph (also called a digraph) is $G = (V, E)$, where

- $V$ is a set of vertices or nodes
- $E \subseteq V \times V$ is set of ordered pairs of vertices called edges
Directed Graph Connectivity Problems

1. Given $G$ and nodes $u$ and $v$, can $u$ reach $v$?
2. Given $G$ and $u$, compute $rch(u)$.
3. Given $G$ and $u$, compute all $v$ that can reach $u$, that is all $v$ such that $u \in rch(v)$.
4. Find the strongly connected component containing node $u$, that is $SCC(u)$.
5. Is $G$ strongly connected (a single strong component)?
6. Compute all strongly connected components of $G$. 

First five problems can be solved in $O(n + m)$ time by adapting BFS/DFS to directed graphs. The last one requires a clever DFS based algorithm.
Directed Graph Connectivity Problems

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DFS in Directed Graphs

**DFS(G)**
- Mark all nodes $u$ as unvisited
- $T$ is set to $\emptyset$
- $\text{time} = 0$
- while there is an unvisited node $u$ do
  - DFS($u$)
- Output $T$

**DFS($u$)**
- Mark $u$ as visited
- $\text{pre}(u) = ++\text{time}$
- for each edge $(u, v)$ in $\text{Out}(u)$ do
  - if $v$ is not marked
    - add edge $(u, v)$ to $T$
    - DFS($v$)
- $\text{post}(u) = ++\text{time}$
Example

A directed graph (also called a digraph) is $G = (V, E)$, where $V$ is a set of vertices or nodes and $E \subseteq V \times V$ is the set of ordered pairs of vertices called edges.
DFS Properties

Generalizing ideas from undirected graphs:

1. **DFS(u)** outputs a directed out-tree **T** rooted at **u**
DFS Properties

Generalizing ideas from undirected graphs:

1. $\text{DFS}(u)$ outputs a directed out-tree $T$ rooted at $u$
2. A vertex $v$ is in $T$ if and only if $v \in \text{rch}(u)$
DFS Properties

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Generalizing ideas from undirected graphs:

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3. For any two vertices \( x, y \) the intervals \([\text{pre}(x), \text{post}(x)]\) and \([\text{pre}(y), \text{post}(y)]\) are either disjoint or one is contained in the other.
4. After initialization of Mark array and data structures: the running time of **DFS(u)** is \( O(k) \) where \( k = \sum_{v \in rch(u)} |\text{Adj}(v)| \).
**DFS Properties**

Generalizing ideas from undirected graphs:

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5. \( \text{DFS}(G) \) takes \( O(m + n) \) time.
DFS Properties

Generalizing ideas from undirected graphs:

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6. Edges added form a *branching*: a forest of out-trees. Output of **DFS**(G) depends on the order in which vertices are considered.
DFS Properties

Generalizing ideas from undirected graphs:

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5. **DFS(G)** takes \( O(m + n) \) time.
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Note: Not obvious whether **DFS(G)** is useful in dir graphs but it is.
DFS Tree

Edges of $G$ can be classified with respect to the DFS tree $T$ as:

1. **Tree edges** that belong to $T$

2. A **forward edge** is a non-tree edges $(x, y)$ such that $\text{pre}(x) < \text{pre}(y) < \text{post}(y) < \text{post}(x)$.

3. A **backward edge** is a non-tree edge $(y, x)$ such that $\text{pre}(x) < \text{pre}(y) < \text{post}(y) < \text{post}(x)$.

4. A **cross edge** is a non-tree edges $(x, y)$ such that the intervals $[\text{pre}(x), \text{post}(x)]$ and $[\text{pre}(y), \text{post}(y)]$ are disjoint.
Types of Edges

- **A**
- **B**
- **C**
- **D**

Directed Edges:
- **Forward** from A to B
- **Backward** from B to A

Undirected Edge:
- **Cross** between C and D
Directed Graph Connectivity Problems

1. Given $G$ and nodes $u$ and $v$, can $u$ reach $v$?
2. Given $G$ and $u$, compute $rch(u)$.
3. Given $G$ and $u$, compute all $v$ that can reach $u$, that is all $v$ such that $u \in rch(v)$.
4. Find the strongly connected component containing node $u$, that is $SCC(u)$.
5. Is $G$ strongly connected (a single strong component)?
6. Compute all strongly connected components of $G$. 
Algorithms via DFS - I

1. Given $G$ and nodes $u$ and $v$, can $u$ reach $v$?
2. Given $G$ and $u$, compute $rch(u)$.

Use $\text{DFS}(G, u)$ to compute $rch(u)$ in $O(n + m)$ time.
Given $G$ and $u$, compute all $v$ that can reach $u$, that is all $v$ such that $u \in rch(v)$. 

Naive: $O(n(n + m))$

Definition (Reverse graph.)
Given $G = (V, E)$, $G_{rev}$ is the graph with edge directions reversed $G_{rev} = (V, E')$ where $E' = \{(y, x) | (x, y) \in E\}$

Compute $rch(u)$ in $G_{rev}!$

Correctness: exercise

Running time: $O(n + m)$ to obtain $G_{rev}$ from $G$ and $O(n + m)$ time to compute $rch(u)$ via DFS. If both $Out(v)$ and $In(v)$ are available at each $v$ then no need to explicitly compute $G_{rev}$. Can do DFS($u$) in $G_{rev}$ implicitly.
Given $G$ and $u$, compute all $v$ that can reach $u$, that is all $v$ such that $u \in rch(v)$.

Naive: $O(n(n + m))$
Given $G$ and $u$, compute all $v$ that can reach $u$, that is all $v$ such that $u \in rch(v)$.

Naive: $O(n(n + m))$

Definition (Reverse graph.)

Given $G = (V, E)$, $G^{rev}$ is the graph with edge directions reversed $G^{rev} = (V, E')$ where $E' = \{(y, x) \mid (x, y) \in E\}$
Given \( G \) and \( u \), compute all \( v \) that can reach \( u \), that is all \( v \) such that \( u \in rch(v) \).

Naive: \( O(n(n + m)) \)

**Definition (Reverse graph.)**

Given \( G = (V, E) \), \( G^{rev} \) is the graph with edge directions reversed

\[ G^{rev} = (V, E') \]  
where \( E' = \{(y, x) \mid (x, y) \in E\} \)

Compute \( rch(u) \) in \( G^{rev} \)!

1. **Correctness:** exercise

2. **Running time:** \( O(n + m) \) to obtain \( G^{rev} \) from \( G \) and \( O(n + m) \) time to compute \( rch(u) \) via DFS. If both \( Out(v) \) and \( In(v) \) are available at each \( v \) then no need to explicitly compute \( G^{rev} \). Can do \( DFS(u) \) in \( G^{rev} \) implicitly.
$\text{SC}(G, u) = \{ v \mid u \text{ is strongly connected to } v \}$
SC(G, u) = \{ v \mid u \text{ is strongly connected to } v \}

Find the strongly connected component containing node u.
That is, compute SCC(G, u).
\( \text{SC}(G, u) = \{ v \mid u \text{ is strongly connected to } v \} \)

1. Find the strongly connected component containing node \( u \).
   That is, compute \( \text{SCC}(G, u) \).

\( \text{SCC}(G, u) = \text{rch}(G, u) \cap \text{rch}(G^{\text{rev}}, u) \)
SC(G, u) = \{v \mid u \text{ is strongly connected to } v\}

1. Find the strongly connected component containing node u. That is, compute SCC(G, u).

SCC(G, u) = rch(G, u) \cap rch(G^{rev}, u)

Hence, SCC(G, u) can be computed with two DFSes, one in G and the other in G^{rev}. Total O(n + m) time.

Why can rch(G, u) \cap rch(G^{rev}, u) be done in O(n) time?
Is $G$ strongly connected?
Is $G$ strongly connected?

Pick arbitrary vertex $u$. Check if $SC(G, u) = V$. 
Find all strongly connected components of $G$. 
Find *all* strongly connected components of $G$.

\[
\text{for each vertex } u \in V \text{ do} \\
\text{find } SC(G, u)
\]
Find all strongly connected components of $G$.

\[
\text{for each vertex } u \in V \text{ do find } SC(G, u)
\]

Running time: $O(n(n + m))$. 

Find all strongly connected components of $G$.

```
for each vertex $u \in V$ do
  find $SC(G, u)$
```

Running time: $O(n(n + m))$.

Q: Can we do it in $O(n + m)$ time?