1. Given an undirected graph \( G = (V, E) \) a matching \( M \) is a set of edges from \( E \) such that no two edges in \( M \) share an end point. A matching is perfect if \( |M| = |V|/2 \), that is, every vertex is matched by \( M \). It turns out that finding maximum matchings and perfect matchings in bipartite graphs\(^1\) is easier than in general graphs. The goal of this problem is to describe an incorrect reduction to point out that the proof of correctness of a reduction involves showing two directions. Here is the incorrect reduction. Given an arbitrary graph \( G = (V, E) \) create a bipartite graph \( G' = (V_1, V_2, E') \) where \( V_1 \) and \( V_2 \) are copies of \( V \). Formally \( V_1 = \{u^{(1)} \mid u \in V\} \) and \( V_2 = \{u^{(2)} \mid u \in V\} \). For every edge \( (u, v) \in E \) we add two edges \( (u^{(1)}, v^{(2)}) \) and \( (v^{(1)}, u^{(2)}) \) in \( E' \).

- Show that \( G \) has a matching of size \( k \) implies that \( G' \) has a matching of size at least \( 2k \).
- Give an example where \( G' \) has a matching of size at least \( 2k \) but \( G \) does not have a matching of size \( k \). *Hint:* Consider \( G \) to be the union of two disjoint triangles.

2. Self-reduction. We focus on decision problems even when the underlying problem we are interested in is an optimization problem. For most problems of interest we can in fact show that a polynomial-time algorithm for the decision problem also implies a polynomial-time algorithm for the corresponding optimization problem. To illustrate this consider the maximum independent set (MIS) problem.

- Suppose you are given a algorithm that given a graph \( H \) and integer \( \ell \) outputs whether \( H \) has an independent set of size at least \( \ell \). Using this algorithm as a black box, describe a polynomial time algorithm that given a graph \( G \) and integer \( k \) outputs an independent set of size \( k \) in \( G \) if it has one. Note that you can use the black box algorithm more than once. *Hint:* What happens if you remove a vertex \( v \) and the independent set size does not decrease? What if it does?
- How would you efficiently find a maximum independent set in a given graph \( G \) using the black box?

3. A cycle \( C \) in a directed graph \( G \) is called a Hamiltonian cycle if it contains all the vertices of \( G \). The Hamiltonian Cycle problem is the following: given \( G \), does \( G \) contain a Hamiltonian cycle? The Longest Path problem is the following: given a directed graph \( G \) and integer \( k \), is there a simple path of length \( k \) in \( G \)? Assuming that you have a black box algorithm for the Longest Path problem describe a polynomial-time algorithm for the Hamiltonian Cycle problem. *Prove* the correctness of your algorithm.

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\(^1\)A graph \( G = (V, E) \) is bipartite if \( V \) can be partitioned into \( V_1 \) and \( V_2 \) such that all edges have one end point in \( V_1 \) and the other in \( V_2 \); that is, \( V_1 \) and \( V_2 \) are independent sets.