True or False? (from last semester's final exam)

1. For each of the following questions, indicate ***every*** correct answer by marking the "Yes" box, and indicate ***every*** incorrect answer by marking the "No" box. **Assume P ≠ NP.** If there is any other ambiguity or uncertainty about an answer, mark the "No" box. For example:

~~Yes~~ (X)	No	$2+2=4$
Yes	~~No~~ (X)	$x+y=5$
Yes	~~No~~ (X)	3SAT can be solved in polynomial time.
~~Yes~~ (X)	No	Jeff is not the Queen of England.

There are 40 yes/no choices altogether. Each correct choice is worth ½ point. Each incorrect choice is worth −¼ point.

(a) Consider an arbitrary language $L \subseteq \{0,1\}^*$. Which of the following statements ***must*** be true?

Yes	No	If L is decidable, then L is infinite.
Yes	No	If L is not decidable, then L is infinite.
Yes	No	If L is the union of two regular languages, then its complement $\overline{L}$ is context-free.
Yes	No	If L is context-free, then its complement $\overline{L}$ is context-free.
Yes	No	If L is finite, then L is context-free.

(b) Consider the following sets of languages over the alphabet $\{0,1\}$:

- $\mathscr{L}_{\text{DTM}}$ is the set of all languages $L \subseteq \{0,1\}^*$ such that L is accepted by at least one deterministic Turing machine.
- $\mathscr{L}_{\text{NTM}}$ is the set of all languages $L \subseteq \{0,1\}^*$ such that L is accepted by at least one non-deterministic Turing machine.

Which of the following statements is/are true? (Recall that $A \subsetneq B$ means "A is a *proper* subset of B".)

Yes	No	$\mathscr{L}_{\text{DTM}} \subsetneq \mathscr{L}_{\text{NTM}}$
Yes	No	$\mathscr{L}_{\text{DTM}} \supsetneq \mathscr{L}_{\text{NTM}}$
Yes	No	$\mathscr{L}_{\text{DTM}} = \mathscr{L}_{\text{NTM}}$

(c) Let $L \subseteq \{0, 1\}^*$ be the language accepted by some nondeterministic Turing machine in time $O(f(n))$. Which of the following statements ***must*** be true?

Yes	No	L is decidable.
Yes	No	Some deterministic Turing machine accepts L in time $O(f(n))$.
Yes	No	Some deterministic Turing machine accepts L in time $O(c^{f(n)})$ for some constant c.
Yes	No	Some deterministic Turing machine accepts L, but there is no guaranteed bound on its running time.
Yes	No	L is not necessarily accepted by any deterministic Turing machine.

(d) Suppose some language $A \in \{0, 1\}^*$ reduces to another language $B \in \{0, 1\}^*$. Which of the following statements ***must*** be true?

Yes	No	A Turing machine that recognizes A can be used to construct a Turing machine that recognizes B.
Yes	No	A is decidable.
Yes	No	If B is decidable then A is decidable.
Yes	No	If A is decidable then B is decidable.
Yes	No	If B is NP-hard then A is NP-hard.

(e) Suppose there is a polynomial-time reduction from problem A to problem B. Which of the following statements ***must*** be true?

Yes	No	Problem B is NP-hard.
Yes	No	A polynomial time algorithm for B can be used to solve A in polynomial time.
Yes	No	If B has no polynomial-time algorithm then neither does A.
Yes	No	If A is NP-hard and B has a polynomial-time algorithm then $\text{P} = \text{NP}$.
Yes	No	If B is NP-hard then A is NP-hard.

(f) Recall the universal language $L_u = \{i\#w \mid \text{Turing machine } M_i \text{ accepts input } w\}$. Which of the following statements about its complement $\overline{L_u} = \{0, 1, \#\}^* \setminus L_u$ are true?

Yes	No	$\overline{L_u}$ is regular.
Yes	No	$\overline{L_u}$ is infinite.
Yes	No	$\overline{L_u}$ is recursive.
Yes	No	$\overline{L_u}$ is recursively enumerable but not recursive.
Yes	No	$\overline{L_u}$ is not recursively enumerable.

(g) Which of the following computational models can be simulated by a three-tape deterministic Turing machine, with at most polynomial slow-down in time, assuming P $\neq$ NP?

Yes	No	A python program
Yes	No	A deterministic Turing machine with 1 tape
Yes	No	A deterministic Turing machine with 100 tapes
Yes	No	A nondeterministic Turing machine with 1 tape
Yes	No	A nondeterministic finite-state automaton (NFA)

(h) Which of the following problems are decidable?

Yes	No	Given an NFA N, is the language $L(N)$ infinite?
Yes	No	Given a context-free grammar G and a string w, is w in the language $L(G)$?
Yes	No	Given an undirected graph G, does G contain a Hamiltonian cycle?
Yes	No	Given a Turing machine M with tape alphabet $\{0, 1, \square\}$ and input word $w \in \{0, 1\}^*$, does M ever change a 1 on its tape to a 0?
Yes	No	Given a Turing machine M with tape alphabet $\{0, 1, \square\}$ and input word $w \in \{0, 1\}^*$, does M ever change a $\square$ on its tape to either 0 or 1?
Yes	No	Given two Turing machines M and M', is there a string w that is accepted by both M and M'?
Yes	No	Given a Turing machine M and a string w, does M accept w after at most $|w|^2$ steps?

NP-hardness

1. Jerry Springer and Maury Povich have decided not to compete with each other over scheduling guests during the next talk-show season. There is only one set of Weird People who either host would consider having on their show. The hosts want to divide the Weird People into two (disjoint) groups - those to appear on Jerry's show, and those to appear on Maury's show. (Neither wants to "recycle" a guest that appeared on the other's show.)

   Both Jerry and Maury have preferences about which Weird People they are particularly interested in. For example, Jerry wants to be sure to get at least one person who fits the category "had extra-terrestrial affair". Thus, on his list of preferences, he writes "w_1 or w_3 or w_{45}", since weird people numbered 1, 3, and 45 are the only ones who fit that description. Jerry has other preferences as well, so he lists those also. Similarly, Maury might like to guarantee that on his show, there is at least one guest who confesses to "really enjoying the pumping lemma". Note that people may fall into any number of different categories, such as the person who enjoys the pumping lemma more than the extra-terrestrial affair they had.

   Jerry and Maury each prepare a list reflecting all of their preferences. Each list contains a collection of statements of the form "(w_i or w_j or w_k)". Show that the problem of assigning weird guests to the two shows such that all preferences of both Jerry and Maury are met, is NP-complete. In particular:

   (a) Show that TSA is NP-hard by reducing no-mixed-clauses-3SAT to TSA, where no-mixed-clauses-3SAT is the problem of finding a satisfying assignment, if one exists, of a given 3CNF formula in which no clause contains both a negated variable and a non-negated variable. (For this part you are given as an assumption that no-mixed-clauses-3SAT is NP-hard.)

   (b) Prove that no-mixed-clauses-3SAT is NP-hard by reducing 3SAT to it.

2. Prove that the following variants of SAT is NP-hard. *[Hint: Describe reductions from 3SAT.]*

   (a) Given a boolean formula F in conjunctive normal form, where *each variable appears in at most three clauses*, determine whether F has a satisfying assignment. *[Hint: First consider the variant where each variable appears in at most* ***five*** *clauses.]*

   (b) Given a boolean formula F in conjunctive normal form *and given one satisfying assignment for F*, determine whether F has at least one other satisfying assignment.

3. Bill Clinton is planning a White House party for his staff. His staff has a hierarchical structure; that is, the supervisor relation forms a directed, rooted, acyclic graph, with Bill at the top, and there is an edge from person i to person j in the graph if and only if i is one of several possible immediate supervisors of person j.

   Vice-president Gore has assigned each White House staff member a "party-hound" rating, which is a nonnegative real number reflecting how likely it is that the person will leave the party wearing a monkey suit and a lampshade. In order to make the party fun for all guests, Bill wants to ensure that if a person i attends, then none of i's immediate supervisors attend also.

   Show that it is NP-hard to determine a guest-list subject to the above constraints, which maximizes the sum of the party-hound ratings of all guests.

   [Hint: This problem can be solved in polynomial time when the input graph is a tree!]

4. Prove that the following problem (which we call MATCH) is NP-hard. The input is a finite set S of strings, all of the same length n, over the alphabet $\{0, 1, 2\}$. The problem is to determine whether there is a string $w \in \{0, 1\}^n$ such that for every string $s \in S$, the strings s and w have the same symbol in at least one position.

   For example, given the set $S = \{01220, 21110, 21120, 00211, 11101\}$, the correct output is TRUE, because the string $w = 01001$ matches the first three strings of S in the second position, and matches the last two strings of S in the last position. On the other hand, given the set $S = \{00, 11, 01, 10\}$, the correct output is FALSE.

   [Hint: Describe a reduction from SAT (or 3SAT)]

5. To celebrate the end of the semester, Professor Jarling want to treat himself to an ice-cream cone, at the *Polynomial House of Flavors*. For a fixed price, he can build a cone with as many scoops as he'd like. Because he has good balance (and because we want this problem to work out), assume that he can balance any number of scoops on top of the cone without it tipping over. He plans to eat the ice cream one scoop at a time, from top to bottom, and doesn't want more than one scoop of any flavor.

   However, he realizes that eating a scoop of bubblegum ice cream immediately after the scoop of potatoes-and-gravy ice cream would be unpalatable; these two flavors clearly should not be placed next to each other in the stack. He has other similar constraints; certain pairs of flavors cannot be adjacent in the stack.

   He'd like to get as much ice cream as he can for the one fee by building the tallest cone possible that meets his flavor-incompatibility constraints. Prove that this problem is NP-hard.

6. At the end of the semester, Professors Erickson and Pitt are forced to solve the following EXAMDESIGN problem. They have a list of problems, and they know for each problem which students will *really enjoy* that problem. They need to choose a subset of problems for the exam such that for each student in the class, the exam includes at least one question that student will really enjoy. On the other hand, the professors do not want to spend the entire summer grading an exam with dozens of questions, so the exam must also contain as few questions as possible. Prove that EXAMDESIGN is NP-hard.

7. Which of the following would resolve the P vs. NP question? Justify each answer with a short sentence or two.

   (a) The construction of a polynomial time algorithm for a problem in NP.

   (b) A polynomial-time reduction from 3SAT to the language $\{0^p \mid p \text{ is prime}\}$.

   (c) A polynomial-time reduction from 3COLOR to VERTEXCOVER.

   (d) The existence of a nondeterministic TM that cannot be simulated by any deterministic TM without time loss.

Turing machines and undecidability

For each of the following problems, either *sketch* an algorithm or prove that the problem is undecidable. Recall that w^R denotes the reversal of string w.

8. Given the encoding $\langle M \rangle$ of a Turing machine M, does M accept $\langle M \rangle^R$?

9. Given the encoding $\langle M \rangle$ of a Turing machine M and a string w, does M accept the string ww^R?

10. Given the encoding $\langle M \rangle$ of a Turing machine M, does M accept any palindrome?

11. Given the encodings $\langle M \rangle$ and $\langle M' \rangle$ of two Turing machines M and M', is there a string w that is accepted by both M and M'?

12. Given the encodings $\langle M \rangle$ and $\langle M' \rangle$ of two Turing machines M and M', is there a string w that is accepted by both M and M'?

13. Given the encoding $\langle M \rangle$ of a Turing machine M and a string w, does M accept w in at most $2^{|w|}$ steps?

14. Given the encoding $\langle M \rangle$ of a Turing machine M, is there a string w such that M moves its head to the right on input w?

$\star$15. Given the encoding $\langle M \rangle$ of a Turing machine M and a string w, does M ever change a symbol on its tape when given input w?