

Program Verification: Lecture 15

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Concurrent Objects in Rewriting Logic

Rewriting logic can model very naturally many different kinds of concurrent systems. We have, for example, seen that Petri nets can be naturally formalized as rewrite theories. The same is true for many other models of concurrency such as CCS, the π -calculus, dataflow, real-time models, and so on.

One of the most useful and important classes of concurrent systems is that of **concurrent object systems**, made out of **concurrent objects**, which encapsulate their own local state and can **interact** with other objects in a variety of ways, including both **synchronous interaction**, and **asynchronous communication by message passing**.

Concurrent Objects in Rewriting Logic (II)

It is of course possible to **represent** a concurrent object system as a rewrite theory with somewhat different modeling styles and adopting different **notational conventions**.

What follows is a particular style of representation that has proved useful and expressive in practice, and that is supported by Full Maude's **object-oriented modules**.

It is also possible to define object-oriented modules in Core Maude using the **conf** attribute to specify an associative commutative multiset union operators as a constructor of configurations of objects and messages; the **frewrite** command then ensures object and message fair executions (see the All About Maude book).

Concurrent Objects in Rewriting Logic (III)

To model a concurrent object system as a rewrite theory, we have to explain two things:

- how the **distributed states** of such a system are equationally axiomatized and modeled by the initial algebra of an equational theory (Σ, E) , and
- how the **concurrent interactions** between objects are **axiomatized by rewrite rules**.

We first explain how the distributed states are equationally axiomatized.

Configurations

Let us consider the key state-building operations in Σ and the equations E axiomatizing the distributed states of concurrent object systems. The concurrent state of an object-oriented system, often called a **configuration**, has typically the structure of a **multiset** made up of **objects** and **messages**.

Therefore, we can view configurations as built up by a binary multiset union operator which we can represent with empty syntax (i.e. juxtaposition) as,

$$_ _ : \mathbf{Conf} \times \mathbf{Conf} \longrightarrow \mathbf{Conf}.$$

Configurations (II)

The operator $_ _$ is declared to satisfy the structural laws of **associativity and commutativity** and to have **identity** `null`. Objects and messages are singleton multiset configurations, and belong to subsorts

Object **Msg** $<$ **Conf**,

so that more complex configurations are generated out of them by multiset union.

Configurations (III)

An **object** in a given state is represented as a term

$$\langle O : C \mid a_1 : v_1, \dots, a_n : v_n \rangle$$

where O is the **object's name** or identifier, C is its **class**, the a_i 's are the names of the object's **attribute identifiers**, and the v_i 's are the corresponding **values**.

The set of all the attribute-value pairs of an object state is formed by repeated application of the binary union operator $_ , _$ which also obeys structural laws of associativity, commutativity, and identity; i.e., the order of the attribute-value pairs of an object is immaterial.

Configurations (IV)

The value of each attribute shouldn't be arbitrary: it should have an appropriate **sort**, dictated by the nature of the attribute.

Therefore, in Full Maude **object classes** can be declared in **class declarations** of the form,

$$\text{class } C \mid a_1 : s_1, \dots, a_n : s_n .$$

where C is the class name, and s_i is the sort required for attribute a_i .

We can illustrate such class declarations by considering three classes of objects, **Buffer**, **Sender**, and **Receiver**.

Configurations (IV)

A **buffer** stores a list of integers in its `q` attribute. Lists of integers are built using an associative list concatenation operator, `_ . _` with identity `nil`, and integers are regarded as lists of length one. The name of the object reading from the buffer is stored in its `reader` attribute; such names belong to a sort `Oid` of **object identifiers**. Therefore, the class declaration for buffers is,

```
class Buffer | q : IntList, reader: Oid .
```

The **sender** and **receiver** objects store an integer in a `cell` attribute that can also be empty (`mt`) and have also a counter (`cnt`) attribute. The sender stores also the name of the receiver in an additional attribute.

Configurations (V)

The counter attribute is used to ensure that messages are received by the receiver in the same order as they are sent by the sender, even though communication between the two parties is asynchronous.

Each time the sender gets a new value from the buffer, it increments its counter. It later uses the current value of the counter to tag the message sent with that value to the receiver.

The receiver only accepts a message whose tag is its current counter. It then increments its counter indicating that it is ready for the next message.

Configurations (VI)

The class declarations are:

```
class Sender | cell: Int?, cnt: Int, receiver: Oid .  
class Receiver | cell: Int?, cnt: Int .
```

where `Int?` is a supersort of `Int` having a new constant `mt`.

In Full Maude one can also give **subclass declarations**, with **subclass** syntax (similar to that of **subsort**) so that all the attributes and rewrite rules of a superclass are **inherited** by a subclass, which can have additional attributes and rules of its own.

Configurations (VII)

The **messages** sent by a sender object have the form,

`(to Z : E from (Y,N))`

where Z is the name of the receiver, E is the number sent, Y is the name of the sender, and N is the value of its counter at the time of the sending.

The syntax of messages is user-definable; it can be declared in Full Maude by message operator declarations. In our example by:

```
msg (to _ : _ from (_,_)) : Oid Int Oid Int -> Msg .
```

Object Rewrite Rules

The associativity and commutativity of a configuration's multiset structure make it very fluid. We can think of it as “soup” in which objects and messages float, so that any objects and messages can at any time come together and participate in a concurrent transition corresponding to a communication event of some kind.

In general, the rewrite rules in R describing the dynamics of an object-oriented system can have the form,

Object Rewrite Rules (II)

$$\begin{aligned} r : \quad & M_1 \dots M_n \langle O_1 : F_1 \mid atts_1 \rangle \dots \langle O_m : F_m \mid atts_m \rangle \\ & \longrightarrow \langle O_{i_1} : F'_{i_1} \mid atts'_{i_1} \rangle \dots \langle O_{i_k} : F'_{i_k} \mid atts'_{i_k} \rangle \\ & \quad \langle Q_1 : D_1 \mid atts''_1 \rangle \dots \langle Q_p : D_p \mid atts''_p \rangle \\ & \quad M'_1 \dots M'_q \\ & \quad \text{if } C \end{aligned}$$

where r is the label, the M s are message expressions, i_1, \dots, i_k are different numbers among the original $1, \dots, m$, and C is the rule's condition.

Object Rewrite Rules (III)

That is, a number of objects and messages can come together and participate in a transition in which some new objects may be created, others may be destroyed, and others can change their state, and where some new messages may be created.

If two or more objects appear in the lefthand side, we call the rule **synchronous**, because it forces those objects to jointly participate in the transition. If there is only one object and at most one message in the lefthand side, we call the rule **asynchronous**.

Object Rewrite Rules (IV)

Three typical rewrite rules involving objects in the `Buffer`, `Sender`, and `Receiver` classes are,

```
rl [read] : < X : Buffer | q: L . E, reader: Y >  
           < Y : Sender | cell: mt, cnt: N >  
=> < X : Buffer | q: L, reader: Y >  
   < Y : Sender | cell: E, cnt: N + 1 >
```

```
rl [send] : < Y : Sender | cell: E, cnt: N, receiver: Z >  
=> < Y : Sender | cell: mt, cnt: N > (to Z : E from (Y,N))
```

```
rl [receive] : < Z : Receiver | cell: mt, cnt: N >  
              (to Z : E from (Y,N))  
=> < Z : Receiver | cell: E, cnt: N + 1 >
```

where `E` and `N` range over `Int`, `L` over `IntList`, `X`, `Y`, `Z` over `Oid`, and `L.E` is a list with last element `E`.

Object Rewrite Rules (V)

Notice that the **read** rule is synchronous and the **send** and **receive** rules asynchronous.

Of course, these rules are applied **modulo** the associativity and commutativity of the multiset union operator, and therefore allow both object synchronization and message sending and receiving events anywhere in the configuration, regardless of the position of the objects and messages.

We can then consider the rewrite theory $\mathcal{R} = (\Sigma, E, R)$ axiomatizing the object system with these three object classes, with R the three rules above (and perhaps other rules, such as one for the receiver to write its contents into another buffer object, that are omitted).

Rewrite Theories in General

It is clear that **rewriting logic has the underlying equational logic as a parameter**: the more general the equational logic, the more general the resulting rewrite theories. For example, we have seen that for order-sorted equational logic rules can have the general form,

$$l : t \longrightarrow t' \Leftarrow \left(\bigwedge_i u_i = u'_i \right) \wedge \left(\bigwedge_j w_j \longrightarrow w'_j \right).$$

It has also become increasingly clear that **frozen operators**, that restrict the rewrites allowed below them, are also very useful in practice.

Rewrite Theories in General (II)

We can illustrate frozen operators with the following nondeterministic choice example (in Maude syntax):

```
mod CHOICE is
  protecting INT .
  sorts Elt MSet .
  subsorts Elt < MSet .
  ops a b c d e f g : -> Elt .
  op _ : MSet MSet -> MSet [assoc comm] .
  op card : MSet -> Int [frozen] .
  eq card(X:Elt) = 1 .
  eq card(X:Elt M:MSet) = 1 + card(M:MSet) .
  rl [choice] : X:MSet Y:MSet => Y:MSet .
endm
```

Rewrite Theories in General (III)

It does not make much sense to rewrite below the cardinality function `card`, because then the multiset whose cardinality we wish to determine becomes a **moving target**.

If `card` had not been declared **frozen**, then the rewrites, $a\ b\ c \longrightarrow b\ c \longrightarrow c$ would induce rewrites, $3 \longrightarrow 2 \longrightarrow 1$, which seems bizarre.

The point is that we think of the kind `[MSet]` as the **state kind** in this example, whereas `[Int]` is the **data kind**. By declaring `card` frozen, we restrict rewrites to the state kind, where they belong.

Rewrite Theories in General (IV)

This leads to the following general definition of a rewrite theory on order-sorted equational logic:

A **rewrite theory** is a 4-tuple, $\mathcal{R} = (\Sigma, E, \phi, R)$, where:

- (Σ, E) is a kind-complete order-sorted equational theory, with, say, kinds K , sorts S , and operations Σ
- $\phi : \Sigma \longrightarrow \mathcal{P}_{fin}(\mathbb{N})$ is a $K^* \times K$ -indexed family of functions assigning to each $f : k_1 \dots k_n \longrightarrow k$ in Σ the finite set $\phi(f) \subseteq \{1, \dots, n\}$ of its **frozen argument positions**
- R is a set of (universally quantified) labeled conditional rewrite rules of the form (with t, t' and the w_k, w'_k pairs of terms of same kind)

$$l : t \longrightarrow t' \Leftarrow \left(\bigwedge_i u_i = u'_i \right) \wedge \left(\bigwedge_j w_j \longrightarrow w'_j \right).$$

Rewrite Theories in General (V)

Given a rewrite theory $\mathcal{R} = (\Sigma, E, \phi, R)$, and given a Σ -term $t \in T_\Sigma(X)$, we call a variable $x \in \text{vars}(t)$ **frozen** in t iff there is a nonvariable position $\alpha \in \mathbb{N}^*$ such that $t/\alpha = f(u_1, \dots, u_i, \dots, u_n)$, with $i \in \phi(f)$, and $x \in \text{vars}(u_i)$. Otherwise, we call $x \in X$ **unfrozen**.

Similarly, given Σ -terms $t, t' \in T_\Sigma(X)$, we call a variable $x \in X$ **unfrozen** in t and t' iff it is unfrozen in both t and t' .

Rewriting Logic in General

Given a rewrite theory $\mathcal{R} = (\Sigma, E, \phi, R)$, the sentences that it proves are universally quantified rewrites of the form, $(\forall X) t \longrightarrow t'$, with $t, t' \in T_{\Sigma, E}(X)_k$, for some kind k , which are obtained by finite application of the following **rules of deduction**:

- Reflexivity. For each $t \in T_{\Sigma}(X)$, $\overline{(\forall X) t \longrightarrow t}$

- Equality.
$$\frac{(\forall X) u \longrightarrow v \quad E \vdash (\forall X)u = u' \quad E \vdash (\forall X)v = v'}{(\forall X) u' \longrightarrow v'}$$

- Congruence. For each $f : k_1 \dots k_n \longrightarrow k$ in Σ , with $\{1, \dots, n\} - \phi(f) = \{j_1, \dots, j_m\}$, with $t_i \in T_\Sigma(X)_{k_i}$, $1 \leq i \leq n$, and with $t'_{j_l} \in T_\Sigma(X)_{k_{j_l}}$, $1 \leq l \leq m$,

$$\frac{(\forall X) t_{j_1} \longrightarrow t'_{j_1} \quad \dots \quad (\forall X) t_{j_m} \longrightarrow t'_{j_m}}{(\forall X) f(t_1, \dots, t_{j_1}, \dots, t_{j_m}, \dots, t_n) \longrightarrow f(t_1, \dots, t'_{j_1}, \dots, t'_{j_m}, \dots, t_n)}$$

- Replacement. For each finite substitution $\theta : X \longrightarrow T_\Sigma(Y)$, with, say, $X = \{x_1, \dots, x_n\}$, and $\theta(x_l) = p_l$, $1 \leq l \leq n$, and for each rule in R of the form,

$$l : (\forall X) t \longrightarrow t' \Leftarrow \left(\bigwedge_i u_i = u'_i \right) \wedge \left(\bigwedge_j w_j \longrightarrow w'_j \right)$$

with $Z = \{x_{j_1}, \dots, x_{j_m}\}$, the set of unfrozen variables in t and t' , then,

$$\frac{\left(\bigwedge_r (\forall Y) p_{j_r} \longrightarrow p'_{j_r} \right) \quad \left(\bigwedge_i (\forall Y) \theta(u_i) = \theta(u'_i) \right) \quad \wedge \quad \left(\bigwedge_j (\forall Y) \theta(w_j) \longrightarrow \theta(w'_j) \right)}{(\forall Y) \theta(t) \longrightarrow \theta'(t')}$$

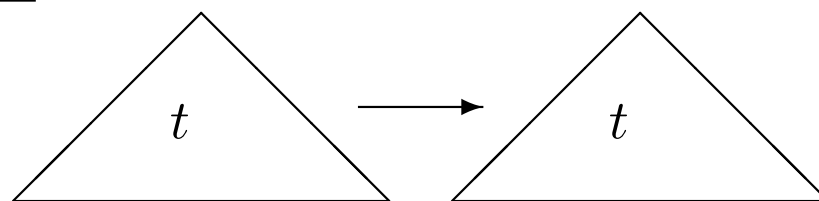
where for $x \in X - Z$, $\theta'(x) = \theta(x)$, and for $x_{j_r} \in Z$, $\theta'(x_{j_r}) = p'_{j_r}$, $1 \leq r \leq m$.

- Transitivity

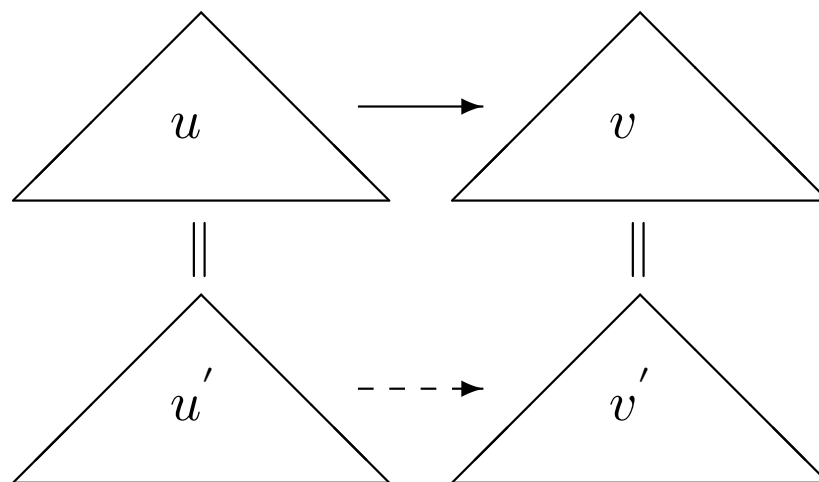
$$\frac{(\forall X) t_1 \longrightarrow t_2 \quad (\forall X) t_2 \longrightarrow t_3}{(\forall X) t_1 \longrightarrow t_3}$$

Rewriting Logic in Pictures

Reflexivity

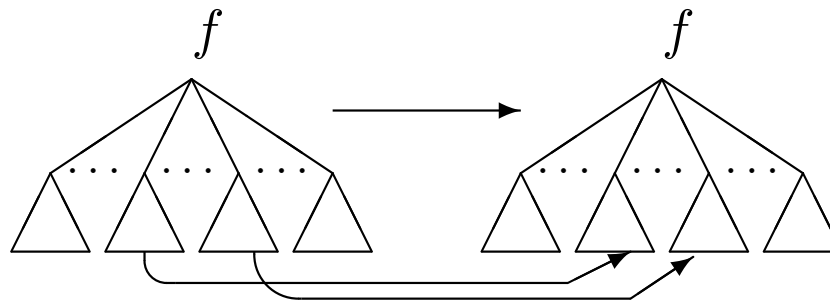


Equality

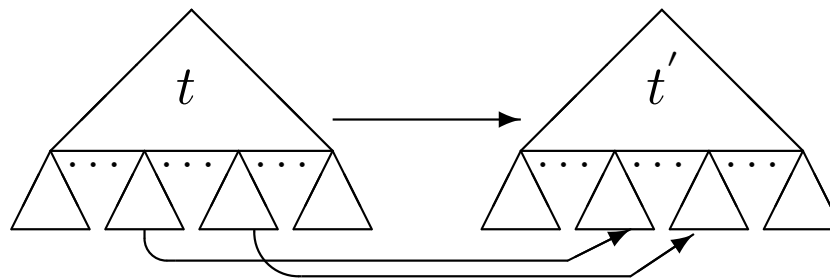


Rewriting Logic in Pictures (II)

Congruence

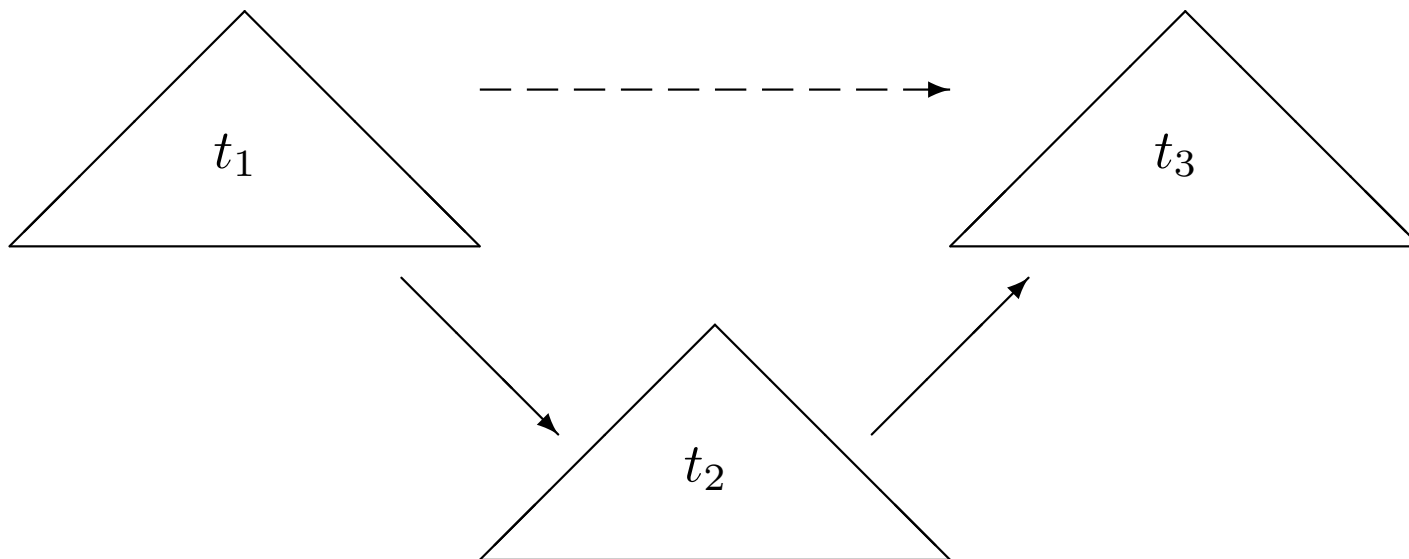


Replacement



Rewriting Logic in Pictures (III)

Transitivity



Computational Meaning of the Inference Rules

Rewriting logic is a **computational logic** to specify concurrent systems. Its inference system allows us to infer **all** the possible finitary concurrent computations of a system specified as a rewrite theory \mathcal{R} as follows:

- Reflexivity is just the possibility of having **idle** transitions
- Equality means that states are equal **modulo** E
- Congruence is a general form of **sideways parallelism**
- Replacement combines an **atomic transition** at the top using a rule with **nested concurrency** in the substitution
- Transitivity is **sequential composition** of concurrent transitions.

Logical and Computational Readings

A rewrite theory $\mathcal{R} = (\Sigma, E, R)$ has two closely related, yet different, **readings**, one computational, and another logical.

Computationally, a rewrite theory specifies a **concurrent system**, whose set of **states** is (a kind in) the initial algebra $T_{\Sigma/E}$. Then, each rewrite rule specifies a parameterized family of **concurrent transitions** in the system.

Logically, a rewrite theory specifies a **logic**, whose set of **formulas** is (a kind in) the initial algebra $T_{\Sigma/E}$. Then, each rewrite rule specifies an **inference rule** in the logic.

Logical and Computational Readings (II)

For example, the **logic of implication** is:

```
mod MINIMALR is sorts SentConstant Formula Configuration .
subsorts SentConstant < Formula < Configuration .
op _→_ : Formula Formula -> Formula .
op empty : -> Configuration .
op __ : Configuration Configuration -> Configuration [assoc comm id: empty] .
vars A B C : Formula .
rl [ax.K] :      empty
              => -----
                 A → (B → A) .
rl [ax.S] :      empty
              => -----
                 (A → B) → ((A → (B → C)) → (A → C)) .
rl [mp] :      (A → B)  A
              => -----
                 B .

endm
```


Logical and Computational Readings (III)

The computational and logical readings are not **mutually exclusive**. Rather, the **same theory** can be regarded computationally, or logically, or **both!**, depending on one's point of view.

For example, logically, our PETRI-MACHINE example is a **linear logic theory** in disguise. It is just a matter of a slight change of syntax, replacing the empty syntax, $_ _$ by that of linear logic's **multiplicative conjunction** operator $_ \otimes _$.

Logical and Computational Readings (IV)

The operator \otimes can be viewed as a form of **resource-conscious** non-idempotent conjunction. Then, the state $a \otimes q \otimes q$ corresponds to having an apple and a quarter and a quarter, which is a strictly better situation than having an apple and a quarter (non-idempotence of \otimes).

Then, in order to get the tensor theory corresponding to PETRI-MACHINE, it is enough to change the arrows into turnstiles, getting the following axioms:

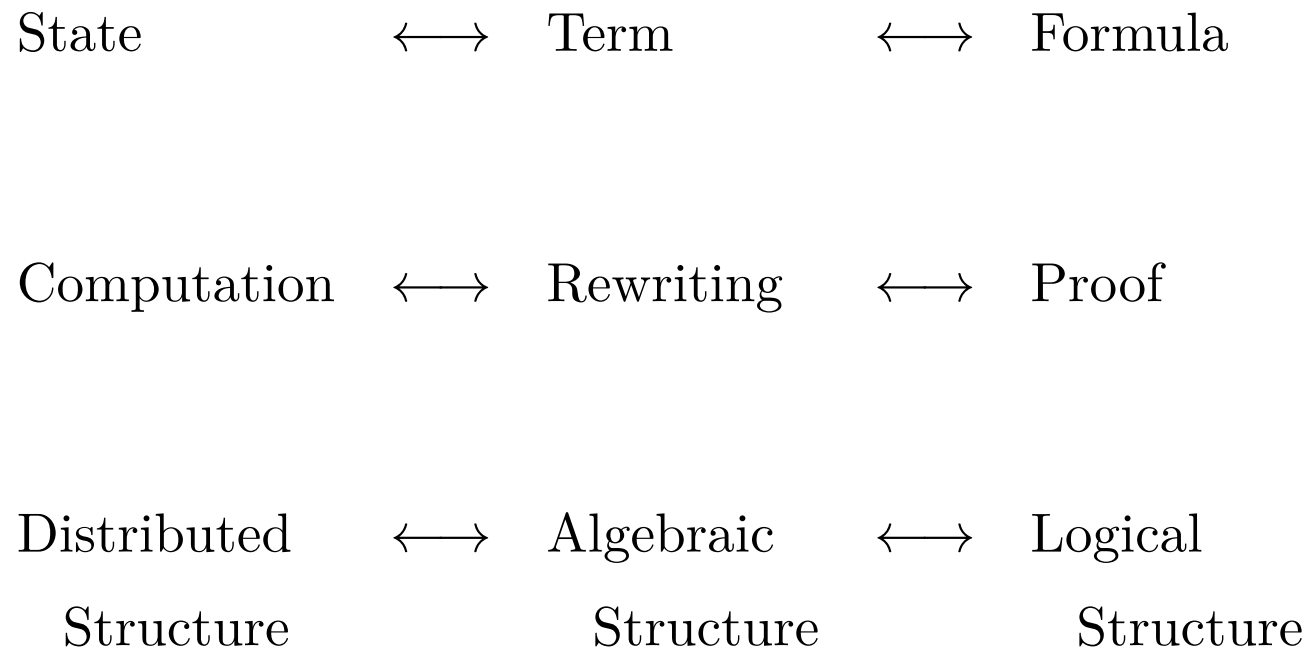
$$\text{buy-c} : \quad \$ \vdash c$$

$$\text{buy-a} : \quad \$ \vdash a \otimes q$$

$$\text{change} : \quad q \otimes q \otimes q \otimes q \vdash \$$$

Logical and Computational Readings (V)

The point is that we have the following equivalences between these two readings:



In particular, **concurrent computations** in our Petri net example **coincide** with linear logic **proofs**.

Exercises

Ex.20.1. Show in detail that, given a Petri net N , the inference system given in pages 39–40 of Lecture 19 to generate all its concurrent computations is **equivalent** to the specialization to the theory $R(N)$ of the general inference system for rewriting logic.