CS 476 Homework #9 Due 10:45am on 4/7

Note: Answers to the exercises listed below should be sent by email to `abir2@illinois.edu`.

1. Consider the unsorted signature $\Sigma_D = \{0, s\}$ called the "Dedekind" signature in STACS. Now consider the following very simple functional module:

```
fmod MOD-2 is
 sort Natural .
 op 0 : -> Natural [ctor] .
 op s : Natural -> Natural [ctor] .
 var x : Natural .
 eq s(s(x)) = x .
endfm
```

Since the equation `s(s(x)) = x` is term-size-decreasing, this module is terminating. It is also very easy to check that it is confluent: you can check the critical pairs themselves (note that there is a non-trivial case of "self-overlap" of the rule with itself at position 1); or you can do so automatically with Maude's the Church-Rosser Checker. As you may have guessed this is one of the simplest possible models of the natural numbers modulo 2. Furthermore, it does have not only an initial algebra $\mathcal{T}_{\Sigma_D/E_2}$, where E_2 is the single equation `s(s(x)) = x`, but, thanks to being convergent, it has also a canonical term algebra $\mathcal{C}_{\Sigma_D/E_2}$.

As a simple exercise to test your understanding of first-order logic you are asked to prove two simple theorems about the initial algebra $\mathcal{T}_{\Sigma_D/E_2}$. Prove that:

(a) $\mathcal{T}_{\Sigma_D/E_2} \models (\forall x)\ (s(x) \neq s)$
(b) $\mathcal{T}_{\Sigma_D/E_2} \models (\forall x)\ (x = 0 \lor x = s(0))$.

Hints. (A) Since we have the isomorphism $\mathcal{T}_{\Sigma_D/E_2} \cong \mathcal{C}_{\Sigma_D/E_2}$ and isomorphic algebras satisfy the same formulas, in (1) and (2) above you can replace $\mathcal{T}_{\Sigma_D/E_2}$ by $\mathcal{C}_{\Sigma_D/E_2}$. (B) Note that: (B.1) all notions related to formula satisfaction in an algebra A are expressed in terms of assignments $a \in [X \to A]$, (B.2) note that the cardinality of the algebra $\mathcal{C}_{\Sigma_D/E_2}$ is exactly 2, and (B.3) note that, even if X is an infinite set of variables, the only variables *that matter* for a given formula φ are those in the finite set Y of variables appearing in φ: that is, for any two assignments $a, a' \in [X \to A]$ for an algebra $\mathcal{A}$ such that $a|_Y = a'|_Y$ we have the equivalence: $\mathcal{A}, a \models \varphi$ iff $\mathcal{A}, a' \models \varphi$.

2. In today's lecture (March 31st 2020) we discussed the following rewrite theory $\mathcal{R} = (\Sigma, E \cup ACU, R)$ where: (1) Σ has a single sort called *Marking* with constants $a, b, c, d, \emptyset$ and a binary multiset union operator $\_\ \_$ with axioms B of associativity, commutativity, and identity element $\emptyset$. (2) R is the Petri net transition: $a\,a\,b\,b \xrightarrow{1} b\,b, c$, where the rule's label 1 is written on top of the arrow for notational convenience. (3) However, there is something else added to this Petri net specification, namely, the set E consisting of the single equation $a\,a = a$. Note that the equational theory $(\Sigma, E \cup ACU)$ is convergent. The canonical forms of the terms in the canonical term algebra $\mathcal{C}_{\Sigma/E,B}$ are all the multisets of the form either $\emptyset$, or $b^m\,c^k$, or $a\,b^m\,c^k$, with $c \geq 0$, $k \geq 0$, where in the case $b^m\,c^k$ we must furthermore have $m + k > 0$ (otherwise $b^m\,c^k$ would become $\emptyset$).

During the lecture, we saw that the above rewrite theory $\mathcal{R}$ is *not* coherent. We also saw that the following two other rewrite rules:

- $a\,b\,b \xrightarrow{2} b\,b, c$
- $a\,b\,b \xrightarrow{3} a\,b\,b, c$

are *necessary* for $\mathcal{R}$ to become coherent, i.e., are required for specific coherence diagrams to hold for some concrete terms. The big question that you are asked to answer is: are rules (1)—(3) *sufficient*, when (2) and (3) are added to $\mathcal{R}$, to make it *ground coherent*, i.e., coherent for every ground term? In summary: you are asked to *prove* that when rules (2) and (3) are added to $\mathcal{R}$, then it becomes ground coherent.

Hint: any ground term in this theory is of the form $a^n\, b^m\, c^k$, where, by convention, $a^0 = b^0 = c^0 = \emptyset$. You can reason by case analysis about for what values of n, m, and k the term $a^n\, b^m\, c^k$ *can be rewritten* by one of the rules (1)–(3). Then, if you can prove by case analysis that in all such cases the coherence property holds no matter which rule happens to be used to rewrite the term $a^n\, b^m\, c^k$, then you will have proved that $\mathcal{R}$ is ground coherent.