Program Verification: Lecture 21

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Deductive Verification of Distributed Systems

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  the terminating state $[u_n]$ satisfies postcondition B.

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Let us see an example of a parametric Hoare triple involving a slight modification of the `CHOICE` module in Lecture 16.

A Hoare Triple for the CHOICE Module

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mod CHOICE is
protecting NAT .
sorts MSet State Pred .
subsorts Nat < MSet .
op __ : MSet MSet -> MSet [ctor assoc comm] .
op {_} : MSet -> State .
op tt : -> Pred [ctor] .
op _=C_ : MSet MSet -> Pred [ctor] .  *** MSet containment
vars U V : MSet . var N : Nat .
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rl [choice] : {U V} => {U} .
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The Hoare triple: $\{\{U\} \mid \top\}$ CHOICE $\{\{N\} \mid N \subseteq U = tt\}$ is *parametric* on U. It states that for each U every final state reachable from $\{U\}$ is a singleton set $\{N\}$ with N in U.

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Skeirik, Stefanescu and Meseguer at UIUC have in turn made reachability logic *rewrite-theory-independent* by defining it for rewrite theories $\mathcal{R}$.

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- directly captures *inductive reasoning* in *any* theory $\mathcal{R}$, unlike Hoare Logic, special rules for loops, etc, *unnecessary*

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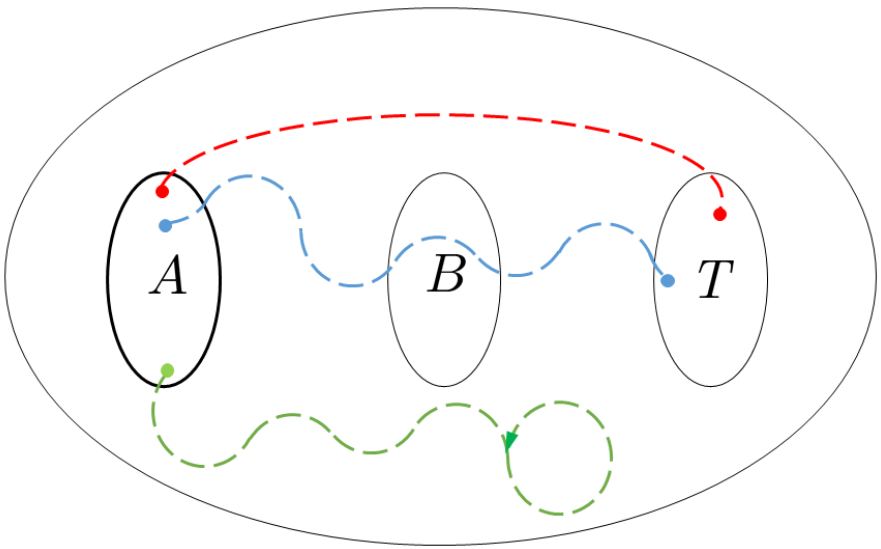

- - - indicates counterex.
- - - satisfies $A \rightarrow^{\circledast} B$
- - - *vacuously* satisfies

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Let $\mathcal{R} = (\Sigma, E \cup B, R)$ be a rewrite theory with good executability conditions, and having a subsignature Ω of constructors and a chosen top sort $State$ of states. Let $\mathcal{C}_{\mathcal{R}}$ denote the canonical reachability model. For a reachability formula $A \longrightarrow^{\circledast} B$ call $Y = vars(A) \cap vars(B)$ its *parameters*.

If $Y = \emptyset$, then we write $\mathcal{R} \models A \longrightarrow^{\circledast} B$ iff for each $[u_0] \in \mathcal{C}_{\mathcal{R},State}$ such that $[u_0] \in [\![A]\!]$ and each *terminating* sequence:

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That is, the parameters Y in $A \longrightarrow^{\circledast} B$ are *universally quantified*, so that $A \longrightarrow^{\circledast} B$ implicitly means: $(\forall Y)\ A \longrightarrow^{\circledast} B$.

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Example. Mutual exclusion from $\langle 0, 0 \rangle$ in `READERS-WRITERS` is the predicate:

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Reachability Logic

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6. Subsumption lets us *discharge* goals where $[\![A]\!] \subseteq [\![B]\!]$

Reachability Logic

Proof Rules

$$\frac{\bigwedge_{j \in J,\, \alpha \in \text{UNIFY}_B(u, l_j)} [\mathcal{A} \cup \mathcal{C},\ \emptyset] \vdash_T (r_j \mid \varphi' \wedge \phi_j)\alpha \longrightarrow^{\circledast} B\alpha}{[\mathcal{A},\ \mathcal{C}] \vdash_T u \mid \varphi \longrightarrow^{\circledast} B}$$

where:

1. $R = \{l_j \to r_j \ \mathit{if}\ \phi_j\}_{j \in J}$
2. $[\![u \mid \varphi']\!] \supseteq [\![u \mid \varphi]\!] \setminus [\![B]\!]$

The STEP Rule

Reachability Logic

Proof Rules

$$\frac{\bigwedge_j [\{u' \mid \varphi' \longrightarrow^{\circledast} \bigvee_j v'_j \mid \psi'_j\} \cup \mathcal{A},\ \emptyset] \ \vdash_T \ v'_j\alpha \mid \varphi \wedge \psi'_j\alpha \longrightarrow^{\circledast} B}{[\{u' \mid \varphi' \longrightarrow^{\circledast} \bigvee_j v'_j \mid \psi'_j\} \cup \mathcal{A},\ \emptyset] \ \vdash_T \ u \mid \varphi \longrightarrow^{\circledast} B}$$

where:

1. $u'\alpha =_B u$
2. $[\![u \mid \varphi]\!] \subseteq [\![(u' \mid \varphi')\alpha]\!]$

The AXIOM Rule

Reachability Logic

Proof Rules

$$\frac{}{[\mathcal{A},\ \mathcal{C}]\ \vdash_T\ A \longrightarrow^{\circledast} B}$$

where: $[\![A]\!] \subseteq [\![B]\!]$

The Subsumption Rule

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Next lecture will illustrate the use of the Maude Reachability Logic tool by means of examples.