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Given unsorted $\Sigma$-algebras $A = (A, \_A)$ and $B = (B, \_B)$, a $\Sigma$-homomorphism $h$ from $A$ to $B$, written $h : A \rightarrow B$, is a function $h : A \rightarrow B$ such that (with $s$ the only sort) we have:

- for each constant $a : \text{nil} \rightarrow s$ in $\Sigma$, $h(a_A) = a_B$ (preservation of constants)

- for each $f : s \rightarrow s. \ldots s \rightarrow s$ in $\Sigma$ and each $(a_1, \ldots, a_n) \in A^n$, we have $h(f_A(a_1, \ldots, a_n)) = f_B(h(a_1), \ldots, h(a_n))$ (preservation of operations)
Examples of Unsorted Homomorphisms

The term algebra $T_{\Sigma_{\text{NAT-MIXFIX}}}$, the natural numbers $\mathbb{N}$, and the natural numbers modulo $n$, $\mathbb{N}_n$ (for any $n \geq 1$) are all $\Sigma_{\text{NAT-PREFIX}}$-algebras (Lectures 2–3). Show that (for any $n$) we have $\Sigma_{\text{NAT-PREFIX}}$-homomorphisms:

$$T_{\Sigma_{\text{NAT-PREFIX}}} \xrightarrow{\text{eval}_{\mathbb{N}}} \mathbb{N} \xrightarrow{\text{rem}_n} \mathbb{N}_n$$

where $\text{eval}_{\mathbb{N}}$ evaluates a term to its value in $\mathbb{N}$, and $\text{rem}_n$ sends each number to its remainder after dividing by $n$. For example, we should have:

- $\text{eval}_{\mathbb{N}}(s(0) + s(0)) = 2$, and
- $\text{rem}_7(23) = 2$.

Show that $\text{eval}_{\mathbb{N}}; \text{rem}_n$ is also a homomorphism, and that we have the identity $\text{eval}_{\mathbb{N}}; \text{rem}_n = \text{eval}_{\mathbb{N}_n}$.
Recall (Lecture 2, pg. 30) the powerset algebra $\mathcal{P}(X)$ over the Boolean signature $\Sigma_{BOOL}$. Let $X$ and $Y$ be any sets, and let $f : X \rightarrow Y$ be any function. Prove in detail that the function

$$f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$$

defined for any $A \subseteq Y$ by: $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$, is a $\Sigma_{BOOL}$-homomorphism. Prove also that if we also have a function $g : Y \rightarrow Z$, then we have the identity $(f; g)^{-1} = g^{-1} \cdot f^{-1}$, and therefore that $g^{-1} \cdot f^{-1} : \mathcal{P}(Z) \rightarrow \mathcal{P}(X)$ is also a $\Sigma_{BOOL}$-homomorphism.
Many-Sorted Homomorphisms

Given (many-sorted) $\Sigma$-algebras $\mathcal{A} = (A, \ldots)$ and $\mathcal{B} = (B, \ldots)$, a $\Sigma$-homomorphism $h$ from $\mathcal{A}$ to $\mathcal{B}$, written $h : A \to B$, is an $S$-indexed family of functions $h = \{h_s : A_s \to B_s\}_{s \in S}$ such that:

- for each constant $a : \text{nil} \to s$, $h_s(a^{\text{nil},s}) = a^{\text{nil},s}$
  (preservation of constants)

- for each $f : w \to s$ with $w = s_1 \ldots s_n$, $n \geq 1$, and each $(a_1, \ldots, a_n) \in A^w$, we have
  $h_s(f^w_{\mathcal{A}}(a_1, \ldots, a_n)) = f^w_{\mathcal{B}}(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))$
  (preservation of operations)
Examples of Many-Sorted Homomorphisms

Recall the module NAT-LIST in Lecture 2, and the two algebras on such a signature, let us call them $\mathcal{A}$ and $\mathcal{B}$, defined on page 34–35 of Lecture 2, namely $\mathcal{A} = \text{lists of natural numbers}$ and $\mathcal{B} = \text{(finite) sets of natural numbers}$. Show that there cannot be any $\Sigma_{\text{NAT-LIST}}$-homomorphism $h : \mathcal{A} \to \mathcal{B}$.

For $\Sigma$ the signature in picture 2.4, consider the first family of algebras for it described in point 1, pages 35–36 of Lecture 2, namely $n$-dimensional vector spaces on the rational, the real, or the complex numbers. Let us be specific and fix the reals. Let $\mathcal{A}$ be the 3-dimensional real vector space, and $\mathcal{B}$ the 2-dimensional real vector space. What is then a $\Sigma$-homomorphism $h : \mathcal{A} \to \mathcal{B}$? Prove that
any such homomorphism $h$ can be completely described by a $2 \times 3$ matrix $M_h$ with real coefficients, so that applying to a 3-dimensionsl vector $\vec{v}$ the homomorphisms $h$, that is, computing $h(\vec{v})$ exactly corresponds to computing the matrix multiplication $\vec{v} \circ M_h$. Generalize this to $A$ and $B$ real vector spaces of arbitrary finite dimensions $n$ and $m$. Generalize it further to rational, resp. complex, vector spaces of any pair of finite dimensions $n$ and $m$.

Now generalize this even further to characterize by means of matrices all $\Sigma$-homomorphims between $\Sigma$-algebras in cases 2–7 in pages 36–38 of Lecture 2, where in case 7 (fuzzy sets) you sould restrict yourself to the fuzzy subsets of finite sets. Give for each of these cases specific examples of $h : A \rightarrow B$ showing how this works and how $h$ is thus applied to specific elements in the corresponding algebra $A$. 
For $\Sigma = ((S, <), F)$ an order-sorted signature, and $A$ and $B$ order-sorted $\Sigma$-algebras, a $\Sigma$-homomorphism $h$ from $A$ to $B$, written $h : A \rightarrow B$, is an $S$-indexed family of functions $h = \{h_s : A_s \rightarrow B_s\}_{s \in S}$ such that:

- $h : A \rightarrow B$ is a many-sorted $(S, F)$-homomorphism; and

- if $[s] = [s']$ and $a \in A_s \cap A_{s'}$, then $h_s(a) = h_{s'}(a)$
  (agreement on data in the same connected component)
Examples of Order-Sorted Homomorphisms

Consider the order-sorted signature $\Sigma$ of the NAT-LIST-II example in Lecture 2, the two algebras on such a signature, let us call them $A$ and $B$, defined on page 40 of Lecture 2, with $A$ case (1), and $B$ case (2). Show that there is exactly one order-sorted $\Sigma$-homomorphism $h : A \rightarrow B$. Describe such a homomorphism $h$ in complete detail. Show that there cannot be any other $\Sigma$-homomorphisms $h' : A \rightarrow B$ with $h \neq h'$. 
If a signature is sensible, then different terms denote different things. In the argot of algebraic specifications, this is expressed by saying that the term algebra has no confusion.

Furthermore, the term algebra is in some sense minimal, since it has only the elements it needs to have to be an algebra: the constants, and the terms needed so that the operations can yield a result; that is why this minimality is expressed saying that it has no junk.

**Note:** In the rest of the course we will always assume that all signatures are sensible.
This minimality means that there is at most one way to map homomorphically the elements of $\mathcal{T}_\Sigma$ to any algebra. And its “no confusion” lack of ambiguity means that such an homomorphic map can always be defined.

For example, it couldn’t be defined for $\Sigma$ the non-sensible signature we showed in pg. 4 of Lecture 3 and the $\Sigma$-algebra $\mathcal{K}$ with: $K_A = \{a\}$, $K_B = \{b\}$, $K_C = \{c\}$, $K_D = \{d, d'\}$, and with $f^{A,B}(a) = b$, $f^{A,C}(a) = c$, $g^{B,D}(b) = d$, and $g^{C,D}(c) = d'$. Indeed, there is no $\Sigma$-homomorphism $h : \mathcal{T}_\Sigma \rightarrow \mathcal{K}$ at all, since $h_D(g(f(a)))$ must be either $d$ or $d'$. If $h_D(g(f(a))) = d$, then $h$ fails to preserve the operation $g : C \rightarrow D$, and if $h_D(g(f(a))) = d'$, then $h$ fails to preserve the operation $g : B \rightarrow D$. 
In summary, the claim is that, if $\Sigma$ is sensible, then for any $\Sigma$-algebra $A$ there is a unique $\Sigma$-homomorphism, say, $\text{eval}_A : T_\Sigma \to A$. This is called the initiality property of $T_\Sigma$. The map $\text{eval}_A$ is the obvious evaluation function, mapping each term $t$ to the result of evaluating it in $A$. $\text{eval}_A$ is defined inductively in the obvious way:

- for a constant $a$ we define $\text{eval}_A(a) = a_A$, and
- for a term $f(t_1, \ldots, t_n)$ we define
  $$\text{eval}_A(f(t_1, \ldots, t_n)) = f_A(\text{eval}_A(t_1), \ldots, \text{eval}_A(t_n)).$$

Let us prove it in detail.

**Theorem.** If $\Sigma$ is a sensible order-sorted signature, then $T_\Sigma$ satisfies the initiality property.
Proof of the Initiality Theorem

**Proof:** For $\mathcal{A}$ any $\Sigma$-algebra Let us first prove the uniqueness of $\text{eval}_{\mathcal{A}}$, and then its existence.

**Proof of uniqueness.** Let us suppose that we have two different homomorphisms $h, h' : T_{\Sigma} \longrightarrow \mathcal{A}$. We can prove that $h = h'$ by induction on the depth of the terms.

For terms of depth 0 let $a$ be a constant in $T_{\Sigma,s}$. That means that there is a sort $s' \leq s$ with an operator declaration $a : \text{nil} \longrightarrow s'$ and therefore, by $h$ and $h'$ being $\Sigma$-homomorphisms we must have $h_s(a) = h'_s(a) = a_{\mathcal{A}}^{\text{nil},s'}$. 
Proof of the Initiality Theorem (II)

Assume that the equality $h = h'$ holds for terms of depth less or equal to $n$, and let $f(t_1, \ldots, t_n) \in T_{\Sigma,s}$ have depth $n + 1$. That means that there is an operator declaration $f : s_1 \ldots s_n \rightarrow s'$ with $s' \leq s$ and $t_i \in T_{\Sigma,s_i}$, $1 \leq i \leq n$. Again, by $h$ and $h'$ being $\Sigma$-homomorphisms we must have:

$$h_s(f(t_1, \ldots, t_n)) =$$

$$= f_{\mathcal{A}}^{s_1 \ldots s_n,s'}(h_{s_1}(t_1), \ldots, h_{s_n}(t_n)) \quad (h \text{ homomorphism and } s \leq s')$$

$$= f_{\mathcal{A}}^{s_1 \ldots s_n,s'}(h'_{s_1}(t_1), \ldots, h'_{s_n}(t_n)) \quad (\text{induction hypothesis})$$

$$= h'_{s}(f(t_1, \ldots, t_n)) \quad (h' \text{ homomorphism and } s \leq s').$$
Proof of Existence. We can both define $\operatorname{eval}_A$ and show that it is a $\Sigma$-homomorphism by induction on the depth of terms. For terms of depth 0, let $a \in T_{\Sigma,s}$ be a constant. That means that there is a sort $s' \leq s$ with an operator declaration $a : \text{nil} \to s'$; we then define $\operatorname{eval}_{A_s}(a) = a^{\text{nil},s'}_A$.

Note that the constant $a$ could be subsort-overloaded (cannot be ad-hoc overloaded, since this is ruled out by $\Sigma$ being sensible) but the above assignment is well-defined (does not depend on the particular declaration $a : \text{nil} \to s'$ chosen), because by our definition of $\Sigma$-algebra the interpretations of all subsort overloaded versions of a constant $a$ must coincide in the algebra $A$. Furthermore, $\operatorname{eval}_A$ preserves constants, so it is a $\Sigma$-homomorphism.
Assume that \( \text{eval}_A \) has already been defined and is a \( \Sigma \)-homomorphism for terms of depth less or equal to \( n \), and let \( f(t_1, \ldots, t_n) \in T_{\Sigma, s} \) be a term of depth \( n + 1 \). That means that there is an operator declaration \( f : s_1 \ldots s_n \rightarrow s' \) with \( s' \leq s \) and \( t_i \in T_{\Sigma, s_i}, 1 \leq i \leq n \). We define
\[
\text{eval}_{A_{s}}(f(t_1, \ldots, t_n)) = f^{s_1 \ldots s_n, s'}_{A}(\text{eval}_{A_{s_1}}(t_1), \ldots, \text{eval}_{A_{s_n}}(t_n)).
\]
Note that, by the induction hypothesis, \( \text{eval}_A \) has already been defined for terms of depth less or equal to \( n \) and is a \( \Sigma \)-homomorphism on those terms.

Note also that, by the Lemma on sensible signatures, for any other \( f : s'_1 \ldots s'_n \rightarrow s'' \) such that \( t_i \in T_{\Sigma, s'_i}, 1 \leq i \leq n \), we must have, \( [s_i] = [s'_i], 1 \leq i \leq n \), and \( [s'] = [s''] \).
Proof of the Initiality Theorem (V)

Since we have \([s_i] = [s'_i], \ 1 \leq i \leq n\), by definition of order-sorted \(\Sigma\)-homomorphism this then forces,

\[
eval_{\mathcal{A}_{s_i}}(t_i) = eval_{\mathcal{A}_{s'_i}}(t_i), \ 1 \leq i \leq n.
\]

Then, by our definition of \(\Sigma\)-algebra, all subsort overloaded operators must agree on common data, so that we have,

\[
f_A^{s_1 \ldots s_n, s'}(eval_{\mathcal{A}_{s_1}}(t_1), \ldots, eval_{\mathcal{A}_{s_n}}(t_n)) = f_A^{s'_1 \ldots s'_n, s''}(eval_{\mathcal{A}_{s'_1}}(t_1), \ldots, eval_{\mathcal{A}_{s'_n}}(t_n)).
\]

Therefore, the definition does not depend on the choice of the subsort overloaded operator. As a consequence, the extension of \(eval_A\) to the step \(n + 1\) is well-defined and, by construction, a \(\Sigma\)-homomorphism, i.e., we have inductively proved the existence of the \(\Sigma\)-homomorphism \(eval_A\). q.e.d.
More on Homomorphisms

Homomorphisms compose. That is, if \( h : A \rightarrow B \) and \( g : B \rightarrow C \) are \( \Sigma \)-homomorphisms, then \( g \circ h = \{ g_s \circ h_s \}_{s \in S} \) is a \( \Sigma \)-homomorphism \( g \circ h : A \rightarrow C \) (Ex. 9.7).

**Notation.** The notation \( g \circ h \) is the most common mathematical notation for function composition and is good to apply the composition to elements, since \( g \circ h(x) = g(h(x)) \) but has the unfortunate drawback of reversing the order of the arrows. Often we will use the alternative notation \( h; g \) which is used for sequential composition in computer science and keeps the order of the arrows.

Identities are homomorphisms. That is, given a \( \Sigma \)-algebra \( \mathcal{A} = (A, \mathcal{A}) \), the family of identity functions \( \text{id}_A = \{ \text{id}_{A_s} \} \) is a \( \Sigma \)-homomorphism \( \text{id}_A : \mathcal{A} \rightarrow \mathcal{A} \).
A $\Sigma$-homomorphism $h : A \to B$ is called an **isomorphism** if there is another $\Sigma$-homomorphism $g : B \to A$ such that $h; g = id_A$ and $g; h = id_B$. We then may use the notation $g = h^{-1}$ and $h = g^{-1}$.

We call a $\Sigma$-homomorphism $h : A \to B$

- **injective** (resp. **surjective**) if for each sort $s \in S$ the function $h_s$ is injective (resp. surjective)

- a **monomorphism** if for any pair of $\Sigma$-homomorphisms $g, q : C \to A$, if $g; h = q; h$ then $g = q$

- an **epimorphism** if for any pair of $\Sigma$-homomorphisms $g, q : B \to C$, if $h; g = h; q$ then $g = q$. 
More on Homomorphisms (III)

For example, if $\mathbb{N}_{bin}$, resp. $\mathbb{N}_{dec}$, denote the natural numbers with 0, successor, and addition in binary, resp. decimal, representation, we have an obvious binary-to-decimal isomorphism $b2d : \mathbb{N}_{bin} \to \mathbb{N}_{dec}$ preserving all operations, whose inverse is the decimal-to-binary isomorphism, $d2b : \mathbb{N}_{bin} \to \mathbb{N}_{dec}$. Of course, $d2b; b2d = id_{\mathbb{N}_{dec}}$, and $b2d; d2b = id_{\mathbb{N}_{bin}}$.

For $\mathbb{N}_n$ the residue classes modulo $n$, the reminder function $\mathbb{N} \xrightarrow{\text{rem}_n} \mathbb{N}_n$ is a surjective homomorphism for $\Sigma$ containing, say, 0, 1, +, $\times$.

Similarly, for $\mathbb{Z}_{dec}$ the integers in decimal notation, the inclusion $j : \mathbb{N}_{dec} \hookrightarrow \mathbb{Z}_{dec}$ is an injective homomorphism preserving all shared operations: 0, 1, +, $\times$, etc.
**Theorem:** All Initial Algebras Are Isomorphic

**Proof:** Suppose $\mathcal{A}$ and $\mathcal{B}$ are $\Sigma$-algebras and both satisfy the initiality property of having a unique $\Sigma$-homomorphism to any other algebra. In particular, we have unique homomorphisms,

$$h : \mathcal{A} \to \mathcal{B} \quad g : \mathcal{B} \to \mathcal{A}$$

and therefore a composed homomorphism

$$h; g : \mathcal{A} \to \mathcal{B} \to \mathcal{A}$$

but we also have the identity homomorphism $id_{\mathcal{A}}$, which by uniqueness forces $h; g = id_{\mathcal{A}}$. Interchanging the role of $\mathcal{A}$ and $\mathcal{B}$ we also get, $g; h = id_{\mathcal{B}}$. q.e.d.
Assignments

Given variables in $X = \{X_s\}$ we will often be interested in assignments (also called valuations) of data elements in a given $\Sigma$-algebra $\mathcal{A} = (A, \_\mathcal{A})$ to those variables. Of course, if $x \in X_s$ then the value, say $a(x)$, assigned to $x$ should be an element of $A_s$. That is the assignments should be well-sorted. All this can be made precise by defining an assignment as an $S$-indexed family of functions,

$$a = \{a_s : X_s \rightarrow A_s\}_{s \in S},$$
denoted $a : X \rightarrow A$.

Often what we want to do with such assignments is to extend them from variables to terms on such variables in the obvious, homomorphic way. That leads us to the algebra $T_{\Sigma(X)}$, and to the following

**Question:** What is a $\Sigma(X)$-algebra?
To properly answer this question we take a little technical detour. Consider a subsignature inclusion $\Omega \subseteq \Sigma$. To simplify life, suppose that $\Omega$ and $\Sigma$ have the same set of sorts $S$ (perhaps with different subsort orderings $\leq_\Omega \subseteq \leq_\Sigma$).

Let now $A = (A, \_A)$ be a $\Sigma$-algebra. We can associate to $A$ an $\Omega$-algebra called its reduct to the signature $\Omega$ and denoted $A|_\Omega$ in a trivial way, just by “forgetting” about the interpretation of operations in $\Sigma$ that are not in $\Omega$. In full detail, we can define $A|_\Omega = (A, \_A|_\Omega)$, where $\_A|_\Omega$ interprets:

- each $a : \text{nil} \rightarrow s$ in $\Omega$, as $\overset{\text{nil},s}{a}_A$

- each $f : w \rightarrow s$ in $\Omega$ as $\overset{w,s}{f}_A$. 


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Example. Let $\Omega$ be the unsorted signature with constant $0$ unary $s$, and binary $+\>$, and let $\Sigma$ be the signature obtained by adding to $\Omega$ the binary operator $\_\ast\>$. Obviously, $\Omega \subseteq \Sigma$. Let $\mathcal{N}$ be the $\Sigma$-algebra whose set of elements is the natural numbers (in any notation you like) and interpreting the symbols as, respectively, zero, successor, addition, and multiplication. What is the algebra $\mathcal{N}|_\Omega$? Well, we just forget about multiplication! That is, it is again the same set of natural numbers, interpreting the symbols of $\Omega$ as, respectively, zero, successor, and addition.

Similarly, for $\mathcal{I}$ the standard interpretation of the integers on $\Sigma$, we just get $\mathcal{I}|_\Omega$ by forgetting about integer multiplication and keeping the rest the same.
To see why the notation \( _A | \Omega \) is eminently natural and suggestive it is useful to recall that if \( \Sigma = ((S, <), F) \) and \( \Omega = ((S, <), G) \), with \( G \subseteq F \), then an interpretation function \( _A \) is an \( S^* \times S \)-indexed function

\[
_A = \{ _A, (w, s) : F_{w,s} \rightarrow [A^w \rightarrow A^s] \} \}
\]

so that \( _A | \Omega \) is just the restriction of such an indexed function to the \( S^* \times S \)-indexed subset \( G = \{ G_{w,s} \} \).
What is a $\Sigma(X)$-Algebra?

In the case of $\Sigma(X)$, it is worth pointing out that there are two useful subsignature inclusions, namely: $\Sigma \subseteq \Sigma(X)$, and $\hat{\Sigma} \subseteq \Sigma(X)$, where, given the $S$-sorted set of variables $X = \{X_s\}_{s \in S}$, $\hat{\Sigma}$ is the $S$-sorted signature with discrete poset structure (many-sorted) and having only the variables of $X$ as constants: $\hat{\Sigma}_{nil,s} = X_s$, and $\hat{\Sigma}_{w,s} = \emptyset$ otherwise.

Now, given a $\Sigma(X)$-algebra $\mathcal{A} = (A, \preceq_A)$ we get two reducts, namely, $\mathcal{A}|_\Sigma = (A, \preceq_{A|_\Sigma})$, and $\mathcal{A}|_{\hat{\Sigma}} = (A, \preceq_{A|_{\hat{\Sigma}}})$. There is nothing special about $\mathcal{A}|_\Sigma$, it is just a $\Sigma$-algebra, but what is $\mathcal{A}|_{\hat{\Sigma}} = (A, \preceq_{A|_{\hat{\Sigma}}})$? $\preceq_{A|_{\hat{\Sigma}}}$ is of course the way the variables in $X$ have been interpreted as constants in $\mathcal{A}$. But this is exactly what an assignment $a : X \rightarrow A$ is by definition.
What is a $\Sigma(X)$-Algebra? (II)

That is, an $\hat{X}$-algebra is exactly a pair $(A, a)$ with $A$ and $S$-indexed set, and $a : X \rightarrow A$ an assignment. So, given our original $\Sigma(X)$-algebra $\mathcal{A} = (A, \_A)$, we have decomposed it into two pieces:

- a $\Sigma$-algebra $\mathcal{A}|_{\Sigma}$, and

- a $\hat{X}$-algebra $\mathcal{A}|_{\hat{X}} = (A, \_A|_{\hat{X}})$, that is, a pair $(A, a)$, with $a$ the assignment $a = \_A|_{\hat{X}}$.

The interesting question now is: can we recover our original $\Sigma(X)$-algebra $\mathcal{A}$ in a unique way from these two pieces? The answer is yes!
What is a $\Sigma(X)$-Algebra? (III)

**Theorem** For $X$ an $S$-sorted set of mutually disjoint variables, also disjoint from an order-sorted signature $\Sigma$ having sorts $S$, any pair $(B, b)$ with $B = (B, \_B)$ a $\Sigma$-algebra and $b : X \rightarrow B$ an assignment determines a unique $\Sigma(X)$-algebra, denoted $B \oplus b$ such that:

1. $(B \oplus b)|_{\Sigma} = B$, and

2. $B \oplus b|_{\hat{X}} = (B, b),$

so that $\_B \oplus b|_{\hat{X}} = b$. Conversely, any $\Sigma(X)$-algebra, $\mathcal{A}$ can be obtained this way, since it satisfies the equation

$$\mathcal{A} = (\mathcal{A}|_{\Sigma}) \oplus (\_\mathcal{A}|_{\hat{X}})$$
What is a $\Sigma(X)$-Algebra? (IV)

**Proof:** Given a pair $(\mathcal{B}, b)$ with $\mathcal{B} = (B, \_B)$ a $\Sigma$-algebra and $b : X \rightarrow B$ an assignment, we define $\mathcal{B} \oplus b$ as the $\Sigma(X)$-algebra having the same $S$-indexed set $B$ as $\mathcal{B}$ and with interpretation function $\_B \oplus b$ such that, restricted to the operations in $\Sigma$ coincides with $\_B$, and for each $s \in S$ it assigns to each $x \in X_s$ the constant $b_s(x)$. It is then obvious that, by construction, $\mathcal{B} \oplus b$ satisfies conditions (1)–(2) in the theorem, that is, that $(\mathcal{B} \oplus b)|_\Sigma = \mathcal{B}$, and $\_B \oplus b|_X = b$.

Uniqueness, that is, the fact that $\mathcal{B} \oplus b$ is the only $\Sigma(X)$-algebra satisfying conditions (1)–(2) is due to the fact that (1) requires that any such $\Sigma(X)$-algebra must have the same $S$-indexed set $B$, and (1) and (2) together uniquely determine the interpretation function $\_B$ on such an algebra.
What is a $\Sigma(X)$-Algebra? (V)

A more detailed justification of this uniqueness claim follows by remarking that if $\Sigma = ((S, <), F)$, then $_-B$ and $b$ are $S^* \times S$-indexed functions

$_-B = \{-B_{w,s} : F_{w,s} \mapsto [B^w \to B^s]\}_{(w,s) \in S^* \times S}$, and

$b = \{b_{w,s} : \hat{X}_{w,s} \mapsto [B^w \to B^s]\}_{(w,s) \in S^* \times S}$, and observing that $\{F_{w,s}\}_{(w,s) \in S^* \times S}$ and $\{\hat{X}_{w,s}\}_{(w,s) \in S^* \times S}$ are disjoint and that $\{F(X)_{w,s}\}_{(w,s) \in S^* \times S} = \{F_{w,s} \uplus \hat{X}_{w,s}\}_{(w,s) \in S^* \times S}$; then one can apply Ex. 9.9.

The converse claim that for any $\Sigma(X)$-algebra, $\mathcal{A}$ we have $\mathcal{A} = (\mathcal{A}|_{\Sigma}) \oplus (\mathcal{A}|-_{\hat{X}})$ follows trivially from the definition of $\mathcal{B} \oplus b$. q.e.d.
What is a $\Sigma(X)$-Algebra? (VI)

The above theorem is very important, because the equation $\mathcal{A} = (\mathcal{A}|_{\Sigma}) \oplus (\neg \mathcal{A}|_{\hat{X}})$ ensures that we can uniquely represent any $\Sigma(X)$-algebra $\mathcal{A}$ as a pair $(\mathcal{A}|_{\Sigma}, \neg \mathcal{A}|_{\hat{X}})$.

On the other hand, any pair $(\mathcal{B}, b)$, with $\mathcal{B} = (B, \neg B)$ a $\Sigma$-algebra and $b : X \rightarrow B$ an assignment, uniquely defines a $\Sigma(X)$-algebra, namely $\mathcal{B} \oplus b$.

Therefore, we can now answer the question: what is a $\Sigma(X)$-algebra? by saying: it is exactly the same thing as a pair $(\mathcal{B}, b)$, with $\mathcal{B} = (B, \neg B)$ a $\Sigma$-algebra and $b : X \rightarrow B$ an assignment. That is, we can freely move back-and-forth between the decomposed representation of such an algebra as a pair $(\mathcal{B}, b)$ or as a unit $\mathcal{B} \oplus b$. 

What is a $\Sigma(X)$-Homomorphism?

This of course, brings us to the next question: what is a $\Sigma(X)$-homomorphism? Since by our previous theorem we know that any $\Sigma(X)$-algebra is always of the form $B \oplus b$ with $B = (B, \_B)$ a $\Sigma$-algebra and $b : X \longrightarrow B$ an assignment, any $\Sigma(X)$-homomorphism is of the form $h : B \oplus b \longrightarrow C \oplus c$ and is by definition an $S$-indexed family of functions $h : B \longrightarrow C$ such that: (i) for each pair of sorts $s, s'$ in the same connected component, $h_s$ and $h_{s'}$ must agree on common data, and (ii) $h$ preserves all the operations in $\Sigma(X)$.

But what does it mean to preserve all the operations on $\Sigma(X)$? Since they are the disjoint union of the operations in $\Sigma$ and the constants in $X$, it exactly means to preserve: (ii).1 the operations in $\Sigma$, and (ii).2 the constants in $X$. 
Now, (i) and (ii).1 mean, by definition, that $h$ defines also a $\Sigma$-homomorphism $h : B \rightarrow C$.

What does (ii).2 mean? Just what it says, of course, that the interpretations of constants in $X$ are preserved; but we can make this meaning more explicit by realizing that those interpretation are given, respectively, by the assignments $b$ and $c$, so what it exactly means is that we have the equality: $b; h = d$.

So, at long last we can aswer our question: a $\Sigma(X)$-homomorphism $h : B \oplus b \rightarrow C \oplus c$ is exactly the same thing as a $\Sigma$-homomorphism $h : B \rightarrow C$ such that $b; h = d$. This is simply depicted in Picture 9.1.
Ex. 9.1. Show that a homomorphism is injective iff it is a monomorphism. Prove that every surjective homomorphism is an epimorphism. Construct an epimorphism that is not surjective.

Ex. 9.2. Show that any many-sorted \( \Sigma \)-homomorphism that is surjective and injective is an isomorphism.

Construct an order-sorted homomorphism that is surjective and injective but is not an isomorphism. Give a sufficient condition on the poset \((S, \leq)\) (more general of course than being a discrete poset, since that is the many-sorted case) so that \(h\) is an isomorphism iff \(h\) is surjective and injective.
Ex. 9.3. Prove that if an algebra $J$ is isomorphic to an initial algebra $I$, then $J$ itself is initial.

Ex. 9.4. Show that the natural numbers in Peano notation (zero and successor) and in base 2 are isomorphic $\Sigma$-algebras (both initial) for $\Sigma$ the signature with one sort $\text{Natural}$ and zero and successor operations.
Ex. 9.5. Show that given a $\Sigma$-homomorphism $h : A \to B$, for each connected component $C$, all the functions 
\[ \{h_s : A_s \to B_s\}_{s \in C} \] “glue together” into a single function $h : A_C \to B_C$.

Ex. 9.6. $S$-indexed functions compose. Show that if $h : A \to B$ and $g : B \to C$ are $S$-indexed functions, then $g \circ h = \{g_s \circ h_s\}_{s \in S}$ is an $S$-indexed function $g \circ h : A \to C$, and that such composition, when defined, is associative, that is if we have $f : C \to D$, then $f \circ (g \circ h) = (f \circ g) \circ h$. Show that if for each $S$-indexed set $A$ we define the family of identity functions $id_A = \{id_{A_s}\}$, then we have, $h \circ id_A = id_B \circ h = h$ for $h : A \to B$. 

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Ex. 9.7. Show all the properties in Ex. 9.6 replacing “$S$-indexed set” by “$\Sigma$-algebra” and “$S$-indexed function” by “$\Sigma$-homomorphism”.

Ex. 9.8. Let $X$ and $Y$ be two disjoint sets, that is, $X \cap Y = \emptyset$, and let $Z$ be a third set. Show that:

1. given any two functions $f : X \to Z$ and $g : Y \to Z$ there is a unique function, denoted $\{f, g\} : X \cup Y \to Z$, such that $i_X; \{f, g\} = f$, and $i_Y; \{f, g\} = g$, where $i_X : X \to X \cup Y$ is defined by $i_X(x) = x$ for each $x \in X$, and likewise, $i_Y(y) = y$ for each $y \in Y$

2. any function $h : X \cup Y \to Z$ is of this form, specifically, $h = \{i_X; h, i_Y; h\}$
3. show that whenever \( X \cap Y \neq \emptyset \), as soon a \( Z \) has two or more elements we can always find functions \( f : X \to Z \) and \( g : Y \to Z \) such that \( \{f, g\} : X \cup Y \to Z \) does not exist.

4. generalize (1) and (2) to the case where \( X \) and \( Y \) are not necessarily disjoint; that is, define for general such \( X \) and \( Y \) a set \( X \oplus Y \) and functions \( i_X : X \to X \oplus Y \), \( i_Y : Y \to X \oplus Y \) such that, given a set \( Z \) and any two functions \( f : X \to Z \) and \( g : Y \to Z \), there is a unique function, denoted \( \{f, g\} : X \oplus Y \to Z \), such that \( i_X ; \{f, g\} = f \), and \( i_Y ; \{f, g\} = g \); show also that any function \( h : X \oplus Y \to Z \) is of this form, specifically, \( h = \{i_X ; h, i_Y ; h\} \).

**Ex. 9.9.** Given two \( S \)-indexed sets, define \( A \cup B = \{A_s \cup B_s\}_{s \in S} \), and \( A \cap B = \{A_s \cap B_s\}_{s \in S} \). Call \( A \) and
B disjoint iff for each \( s \in S \) we have \( A_s \cap B_s = \emptyset \). Generalize all the results of the previous exercise replacing “set” by “\( S \)-indexed set” to get, given \( f : A \to C \) and \( g : B \to C \) a unique \( S \)-indexed family of functions \( \{ f, g \} : A \cup B \to C \) such that \( i_A; \{ f, g \} = f \), and \( i_B; \{ f, g \} = g \). Do the same, defining an appropriate construction of \( A \oplus B \), in the more general case in which \( A \) and \( B \) are not necessarily disjoint.

**Ex.9.10.** Let \( \Sigma \) be the signature:

\begin{verbatim}
sort Nat .
op 0 : -> Nat .
op s : Nat -> Nat .
\end{verbatim}

And let \( A = \{ a, b, c \} \). How many different nonisomorphic \( \Sigma \)-algebra structures can be defined on the set \( A \)? This problem can be stated more precisely as follows. If you solved **Ex.2.2**, you already know the set, say \( Alg_\Sigma(A) \), of all
the $\Sigma$-algebras on $A$. Show that isomorphism is an equivalence relation, say $\cong$, on $Alg_\Sigma(A)$. What you then need to do is to determine the cardinality (number of elements) in the quotient set $Alg_\Sigma(A)/\cong$.

**Ex. 9.11.** Let $\Sigma$ be a sensible order-sorted signature. We say that a $\Sigma$-algebra $A$ has the unique homomorphism property iff for any $\Sigma$-algebra $B$, either there is no homomorphism $A \rightarrow B$, or there is only one. Give a condition on $eval_A : T_\Sigma \rightarrow A$ so that $A$ has the unique homomorphism property if and only if $eval_A$ satisfies that condition. If you cannot give a necessary and sufficient condition, try to give a nontrivial sufficient condition (e.g., requiring $eval_A$ to be an isomorphism would be a trivial sufficient condition because of **Ex. 9.3**).