Program Verification: Lecture 23

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Model checking of invariants and LTL properties is very useful. But it has some limitations:

1. Explicit-state model checking algorithms can only deal with finite sets of reachable states.
2. Even if an equational abstraction can be used to make the set of reachable states finite, the set of abstracted initial states of interest may be infinite.
3. More generally, state infinity can block the use of explicit-state model checking in two different ways: the number of states reachable from a given state is infinite, and the number of initial states is infinite.

This suggests two other options: (1) symbolic model checking (automatic) and (2) deductive methods based on theorem proving (more general). We will explore logics for option (2) in this lecture.
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Deductive Verification of Distributed Systems

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Hoare Logic

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- for each state \([u_0] \in \mathcal{T_{\mathcal{R},\text{State}}}\), if \([u_0]\) satisfies the precondition \( A \), then,
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- for each state $[u_0] \in T_{\mathcal{R},State}$, if $[u_0]$ satisfies the precondition $A$, then,
- for each terminating sequence of transitions

$$[u_0] \rightarrow_\mathcal{R} [u_1] \ldots [u_{n-1}] \rightarrow_\mathcal{R} [u_n]$$
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$$\lbrack u_0 \rbrack \mathcal{R} \lbrack u_1 \rbrack \ldots \mathcal{R} \lbrack u_{n-1} \rbrack \mathcal{R} \lbrack u_n \rbrack$$

the terminating state $\lbrack u_n \rbrack$ satisfies postcondition $B$. 


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What formulas $A$ and $B$ shall we use in a Hoare triple \{\!\!A\!\!\} \\mathcal{R} \\{\!\!B\!\!\}?$ Assuming $\mathcal{R} = (\Sigma, B, R)$ has constructors $\Omega$, we can use *pattern predicates* of the form $u \mid \varphi$ where $u$ is an $\Omega$-term of sort $State$ and $\varphi$ is a $\Sigma$-condition.
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$$[[u \mid \varphi]] = \{[u\rho]_B \mid \rho \in [X \to T_\Omega]\land E \cup B \models \varphi\rho\}.$$
What formulas \( A \) and \( B \) shall we use in a Hoare triple \( \{ A \} \mathcal{R} \{ B \} \)? Assuming \( \mathcal{R} = (\Sigma, B, R) \) has constructors \( \Omega \), we can use pattern predicates of the form \( u \mid \varphi \) where \( u \) is an \( \Omega \)-term of sort \( State \) and \( \varphi \) is a \( \Sigma \)-condition. Then \( u \mid \varphi \) denotes the set of its ground instance states:

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Let \( Y = \text{vars}(A) \cap \text{vars}(B) \). Call \( Y \) the parameters of the Hoare triple \( \{ A \} \mathcal{R} \{ B \} \).
Pattern Predicates and Parameters

What formulas $A$ and $B$ shall we use in a Hoare triple
\{ $A$ \} $R$ \{ $B$ \}? Assuming $R = (\Sigma, B, R)$ has constructors $\Omega$, we can use \textit{pattern predicates} of the form $u \mid \varphi$ where $u$ is an $\Omega$-term of sort $\text{State}$ and $\varphi$ is a $\Sigma$-condition. Then $u \mid \varphi$ denotes the set of its ground instance states:

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Let $Y = \text{vars}(A) \cap \text{vars}(B)$. Call $Y$ the \textit{parameters} of the Hoare triple $\{ A \} R \{ B \}$. Such a triple is in fact \textit{universally quantified} on its parameters. That is, $\{ A \} R \{ B \}$ implicitly means:

$$\forall Y \{ A \} R \{ B \}.$$
What formulas \( A \) and \( B \) shall we use in a Hoare triple \( \{ A \} \mathcal{R} \{ B \} \)? Assuming \( \mathcal{R} = (\Sigma, B, R) \) has constructors \( \Omega \), we can use *pattern predicates* of the form \( u \mid \varphi \) where \( u \) is an \( \Omega \)-term of sort \( \text{State} \) and \( \varphi \) is a \( \Sigma \)-condition. Then \( u \mid \varphi \) denotes the set of its ground instance states:

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Let us see an example of a parametric Hoare triple involving a slight modification of the CHOICE module in Lecture 16.
mod CHOICE is
protecting NAT.
sorts MSet State Pred.
subsorts Nat < MSet.

op __ : MSet MSet -> MSet [ctor assoc comm].

op { } : MSet -> State.

op tt : -> Pred [ctor].

op _=C_ : MSet MSet -> Pred [ctor]. *** MSet containment

vars U V : MSet. var N : Nat.

eq U =C U = tt.

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rl [choice] : {U V} => {U}.

endm
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The Hoare triple: \{\{M\} | \top\} CHOICE \{\{N\} | N \subseteq M = tt\} is *parametric* on \(M\).
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parametric on \(M\). It states that for each \(M\) every final state reachable from \{M\} is a singleton set \{N\} with \(N\) in \(M\).
From Hoare Logic to Reachability Logic

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special case of a Hoare triple of the form $\{A'(p)\} R_L \{B'\}$, where
$L$ is the imperative programming language, and $R_L$ is the
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Skeirik, Stefanescu and Meseguer at UIUC have in turn made reachability logic rewrite-theory-independent by defining it for rewrite theories $\mathcal{R}$. 
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- considers formulas $A \longrightarrow^\ast B$ where $A$ is a *pattern predicate*, and $B$ a *disjunction of pattern predicates*.

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u \models \phi \longrightarrow^\ast \bigvee_{i} v_i \models \psi_i\]
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- a generalization of Hoare Logic partial correctness, i.e., $A \xrightarrow{\diamond} B$ generalizes $\{A\} \mathcal{R} \{B\}$
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- a generalization of Hoare Logic *partial correctness*, i.e., $A \rightarrow^\star B$ generalizes $\{A\}\mathcal{R}\{B\}$
- directly captures *inductive reasoning* in any theory $\mathcal{R}$, unlike Hoare Logic, special rules for loops, etc, *unnecessary*
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Reachability Logic
Sequents

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2. $p$ is infinite
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\(-\-\-\) indicates counterex.
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Reachability Logic
Precise Definition

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If $Y = \emptyset$, then we write $\mathcal{R} \models A \rightarrow^\ast B$ iff for each $[u_0] \in \mathcal{C}_R,State$ such that $[u_0] \in [A]$ and each terminating sequence:

$$[u_0] \rightarrow_{\mathcal{C}_R} [u_1] \ldots [u_{n-1}] \rightarrow_{\mathcal{C}_R} [u_n]$$
Let $\mathcal{R} = (\Sigma, E \cup B, R)$ be a rewrite theory with good executability conditions, and having a subsignature $\Omega$ of constructors and a chosen top sort $State$ of states. Let $C_R$ denote the canonical reachability model. For a reachability formula $A \rightarrow^\ast B$ call $Y = \text{vars}(A) \cap \text{vars}(B)$ its parameters.

If $Y = \emptyset$, then we write $\mathcal{R} \models A \rightarrow^\ast B$ iff for each $[u_0] \in C_R, State$ such that $[u_0] \in \llbracket A \rrbracket$ and each terminating sequence:

$$[u_0] \rightarrow_{C_R} [u_1] \ldots [u_{n-1}] \rightarrow_{C_R} [u_n]$$

there exist $j$, $0 \leq j \leq n$ such that $[u_j] \in \llbracket B \rrbracket$. 
Reachability Logic
Precise Definition

Let $\mathcal{R} = (\Sigma, E \cup B, R)$ be a rewrite theory with good executability conditions, and having a subsignature $\Omega$ of constructors and a chosen top sort $State$ of states. Let $C_\mathcal{R}$ denote the canonical reachability model. For a reachability formula $A \rightarrow^\ast B$ call $Y = \text{vars}(A) \cap \text{vars}(B)$ its parameters.

If $Y = \emptyset$, then we write $\mathcal{R} \models A \rightarrow^\ast B$ iff for each $[u_0] \in C_\mathcal{R}_{State}$ such that $[u_0] \in [A]$ and each terminating sequence:

$$[u_0] \rightarrow_{C_\mathcal{R}} [u_1] \ldots [u_{n-1}] \rightarrow_{C_\mathcal{R}} [u_n]$$

there exist $j$, $0 \leq j \leq n$ such that $[u_j] \in [B]$.

If $Y \neq \emptyset$, then we write $\mathcal{R} \models A \rightarrow^\ast B$ iff
Reachability Logic
Precise Definition

Let \( \mathcal{R} = (\Sigma, E \cup B, R) \) be a rewrite theory with good executability conditions, and having a subsignature \( \Omega \) of constructors and a chosen top sort \( \text{State} \) of states. Let \( \mathcal{C}_R \) denote the canonical reachability model. For a reachability formula \( A \longrightarrow^\ast B \) call \( Y = \text{vars}(A) \cap \text{vars}(B) \) its parameters.

If \( Y = \emptyset \), then we write \( \mathcal{R} \models A \longrightarrow^\ast B \) iff for each \( [u_0] \in \mathcal{C}_R,\text{State} \) such that \( [u_0] \in \llbracket A \rrbracket \) and each terminating sequence:

\[
[u_0] \rightarrow_{\mathcal{C}_R} [u_1] \ldots [u_{n-1}] \rightarrow_{\mathcal{C}_R} [u_n]
\]

there exist \( j, 0 \leq j \leq n \) such that \( [u_j] \in \llbracket B \rrbracket \).

If \( Y \neq \emptyset \), then we write \( \mathcal{R} \models A \longrightarrow^\ast B \) iff for each \( \rho \in [Y \rightarrow T_\Omega] \) we have \( \mathcal{R} \models A\rho \longrightarrow^\ast B\rho \).
Reachability Logic
Precise Definition

Let \( \mathcal{R} = (\Sigma, E \cup B, R) \) be a rewrite theory with good executability conditions, and having a subsignature \( \Omega \) of constructors and a chosen top sort \textit{State} of states. Let \( \mathcal{C}_\mathcal{R} \) denote the canonical reachability model. For a reachability formula \( A \xrightarrow{\ast} B \) call \( Y = \text{vars}(A) \cap \text{vars}(B) \) its \textit{parameters}.

If \( Y = \emptyset \), then we write \( \mathcal{R} \models A \xrightarrow{\ast} B \) iff for each \( [u_0] \in \mathcal{C}_\mathcal{R, State} \) such that \( [u_0] \in \llbracket A \rrbracket \) and each \textit{terminating} sequence:

\[
[u_0] \xrightarrow{\mathcal{C}_\mathcal{R}} [u_1] \ldots [u_{n-1}] \xrightarrow{\mathcal{C}_\mathcal{R}} [u_n]
\]

there exist \( j, 0 \leq j \leq n \) such that \( [u_j] \in \llbracket B \rrbracket \).

If \( Y \neq \emptyset \), then we write \( \mathcal{R} \models A \xrightarrow{\ast} B \) iff for each \( \rho \in [Y \rightarrow T_\Omega] \) we have \( \mathcal{R} \models A\rho \xrightarrow{\ast} B\rho \).

That is, the parameters \( Y \) in \( A \xrightarrow{\ast} B \) are \textit{universally quantified},

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Let $\mathcal{R} = (\Sigma, E \cup B, R)$ be a rewrite theory with good executability conditions, and having a subsignature $\Omega$ of constructors and a chosen top sort $\text{State}$ of states. Let $\mathcal{C}_\mathcal{R}$ denote the canonical reachability model. For a reachability formula $A \rightarrow^* B$ call $Y = \text{vars}(A) \cap \text{vars}(B)$ its parameters.

If $Y = \emptyset$, then we write $\mathcal{R} \models A \rightarrow^* B$ iff for each $[u_0] \in \mathcal{C}_\mathcal{R},\text{State}$ such that $[u_0] \in \llbracket A \rrbracket$ and each terminating sequence:

$$[u_0] \rightarrow_{\mathcal{C}_\mathcal{R}} [u_1] \ldots [u_{n-1}] \rightarrow_{\mathcal{C}_\mathcal{R}} [u_n]$$

there exist $j$, $0 \leq j \leq n$ such that $[u_j] \in \llbracket B \rrbracket$.

If $Y \neq \emptyset$, then we write $\mathcal{R} \models A \rightarrow^* B$ iff for each $\rho \in \llbracket Y \rightarrow T_\Omega \rrbracket$ we have $\mathcal{R} \models A\rho \rightarrow^* B\rho$.

That is, the parameters $Y$ in $A \rightarrow^* B$ are universally quantified, so that $A \rightarrow^* B$ implicitly means: $(\forall Y) A \rightarrow^* B$. 
Q: How is a Hoare triple $\{A\} R \{B\}$ expressed in reachability logic?

A: as the formula $A \rightarrow \Box (B \land T)$, with $J T K$ the terminating states.

Q: How is a reachability logic sequent $A \rightarrow \Box B$ expressed in linear temporal logic?

A: as the LTL formula $A \rightarrow (\Box \text{enabled}) \lor \Diamond B$.

Example. For \textsc{Choice}, the formula $\{M\} |\top \rightarrow \Box \{M'\} |M' \subseteq M = \text{tt}$ is parametric on $M$. It states that for each $M$ every state reachable from $\{M\}$ is a submultiset $M'$ of $M$. Note that this reachability property cannot be expressed by a Hoare triple.
Q: How is a Hoare triple \( \{A\} \mathcal{R} \{B\} \) expressed in reachability logic?

A: as the formula \( A \rightarrow^{\circledast} (B \land T) \), with \([T]\) the terminating states.
Q: How is a Hoare triple \( \{A\} \mathcal{R} \{B\} \) expressed in reachability logic?

A: as the formula \( A \rightarrow^\ast (B \land T) \), with \([T]\) the terminating states.

Q: How is a reachability logic sequent \( A \rightarrow^\ast B \) expressed in linear temporal logic?

Example. For CHOICE, the formula \( \{M\} |\top \rightarrow^\ast \{M'\} |M'\subseteq M \) is parametric on \( M \). It states that for each \( M \) every state reachable from \( \{M\} \) is a submultiset \( M' \) of \( M \). Note that this reachability property cannot be expressed by a Hoare triple.
Q: How is a Hoare triple $\{A\}R\{B\}$ expressed in reachability logic?

A: as the formula $A \rightarrow^\star (B \land T)$, with $\lceil T \rceil$ the terminating states.

Q: How is a reachability logic sequent $A \rightarrow^\star B$ expressed in linear temporal logic?

A: as the LTL formula $A \rightarrow (\lozenge enabled) \lor \diamond B$. 

Example. For CHOICE, the formula $\{M\} | \top \rightarrow^\star \{M\}' | M'$ is parametric on $M$. It states that for each $M$ every state reachable from $\{M\}$ is a submultiset $M'$ of $M$.

Note that this reachability property cannot be expressed by a Hoare triple.
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Q: How is a reachability logic sequent \( A \rightarrow^\ast B \) expressed in linear temporal logic?

A: as the LTL formula \( A \rightarrow (\Box enabled) \lor \Diamond B \).

**Example.** For CHOICE, the formula

\[
\{M\} \mid \top \rightarrow^\ast \{M'\} \mid M' \subseteq M = tt
\]
Q: How is a Hoare triple $\{A\} \mathcal{R} \{B\}$ expressed in reachability logic?

A: as the formula $A \mathrel{-\rightarrow} \mathcal{X} (B \land T)$, with $[T]$ the terminating states.

Q: How is a reachability logic sequent $A \mathrel{-\rightarrow} \mathcal{X} B$ expressed in linear temporal logic?

A: as the LTL formula $A \rightarrow (\Box enabled) \lor \Diamond B$.

Example. For $\text{CHOICE}$, the formula

$$
\{M\} \mid \top \mathrel{-\rightarrow} \mathcal{X} \{M'\} \mid M' \subseteq M = tt
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is parametric on $M$. 
Q: How is a Hoare triple $\{A\} \mathcal{R} \{B\}$ expressed in reachability logic?

A: as the formula $A \longrightarrow^\ast (B \land T)$, with $\lceil T \rceil$ the terminating states.

Q: How is a reachability logic sequent $A \longrightarrow^\ast B$ expressed in linear temporal logic?

A: as the LTL formula $A \rightarrow (\Box enabled) \lor \Diamond B$.

Example. For CHOICE, the formula

$$\{M\} | \top \longrightarrow^\ast \{M'\} | M' \subseteq M = tt$$

is parametric on $M$. It states that for each $M$ every state reachable from $\{M\}$ is a submultiset $M'$ of $M$. 


Q: How is a Hoare triple $\{A\} R \{B\}$ expressed in reachability logic?

A: as the formula $A \rightarrow^\ast (B \land \llbracket T \rrbracket)$, with $\llbracket T \rrbracket$ the terminating states.

Q: How is a reachability logic sequent $A \rightarrow^\ast B$ expressed in linear temporal logic?

A: as the LTL formula $A \rightarrow (\square enabled) \lor \Diamond B$.

Example. For CHOICE, the formula

$$\{M\} | \top \rightarrow^\ast \{M'\} \mid M' \subseteq M = tt$$

is parametric on $M$. It states that for each $M$ every state reachable from $\{M\}$ is a submultiset $M'$ of $M$. Note that this reachability property cannot be expressed by a Hoare triple.
Consider the readers and writers example (Lecture 18):

mod READERS-WRITERS is
  protecting NAT.
  sort State.
  op <_,_> : Nat Nat -> State [ctor].
  vars R W : Nat.
  rl < 0, 0 > => < 0, s(0) >.
  rl < R, s(W) > => < R, W >.
  rl < R, 0 > => < s(R), 0 >.
  rl < s(R), W > => < R, W >.
endm

Q: How can we express its mutual exclusion invariant as a reachability formula $A \rightarrow \Box B$?

A: Since:
  (i) $A \rightarrow \Box B$ just means $A \rightarrow (□ \text{enabled}) \lor \lozenge B$, and
  (ii) READERS-WRITERS is a never terminating rewrite theory, all formulas $A \rightarrow \Box B$ are satisfied!! So we cannot!! (Paradox!!).
Consider the readers and writers example (Lecture 18):

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rl < 0, 0 > => < 0, s(0) > .
rl < R, s(W) > => < R, W > .
rl < R, 0 > => < s(R), 0 > .
rl < s(R), W > => < R, W > .
endm

Q: How can we express its mutual exclusion invariant as a reachability formula \( A \rightarrow \downarrow B \)?

A: Since:
(i) \( A \rightarrow \downarrow B \) just means \( A \rightarrow (\Box \text{enabled}) \lor \Diamond B \), and
(ii) READERS-WRITERS is a never terminating rewrite theory, all formulas \( A \rightarrow \downarrow B \) are satisfied!! So we cannot!! (Paradox!!)
The Invariant Paradox

Consider the readers and writers example (Lecture 18):

mod READERS-WRITERS is
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rl < 0, 0 > => < 0, s(0) > .
rl < R, s(W) > => < R, W > .
rl < R, 0 > => < s(R), 0 > .
rl < s(R), W > => < R, W > .
endm

Q: How can we express its *mutual exclusion* invariant as a
reachability formula $A \rightarrow^{\ast} B$?
The Invariant Paradox

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    rl < s(R), W > => < R, W > .
    endm

Q: How can we express its *mutual exclusion* invariant as a reachability formula $A \longrightarrow^\ast B$?

A: Since:
The Invariant Paradox

Consider the readers and writers example (Lecture 18):

\[
\text{mod READERS-WRITERS is}
\]
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\text{protecting NAT .}
\]
\[
\text{sort State .}
\]
\[
\text{op \( <\_,\_\> : \text{Nat Nat} \rightarrow \text{State} \) [ctor] . --- readers/writers}
\]
\[
\text{vars R W : \text{Nat} .}
\]
\[
\text{rl \( < 0, 0 > \Rightarrow < 0, s(0) > \) .}
\]
\[
\text{rl \( < R, s(W) > \Rightarrow < R, W > \) .}
\]
\[
\text{rl \( < R, 0 > \Rightarrow < s(R), 0 > \) .}
\]
\[
\text{rl \( < s(R), W > \Rightarrow < R, W > \) .}
\]
\[
\text{endm}
\]

Q: How can we express its \textit{mutual exclusion} invariant as a reachability formula \( A \rightarrow^{\ast} B \)?

A: Since: (i) \( A \rightarrow^{\ast} B \) just means \( A \rightarrow (\square \text{enabled}) \lor \Diamond B \), and
The Invariant Paradox

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Q: How can we express its *mutual exclusion* invariant as a reachability formula $A \longrightarrow^{\ast} B$?

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  rl < R, 0 > => < s(R), 0 > .
  rl < s(R), W > => < R, W > .
endm```

**Q:** How can we express its *mutual exclusion* invariant as a reachability formula \( A \longrightarrow^{\ast} B \)?

**A:** Since: (i) \( A \longrightarrow^{\ast} B \) just means \( A \rightarrow (\Box \text{enabled}) \lor \Diamond B \), and (ii) READERS–WRITERS is a *never terminating* rewrite theory, all formulas \( A \longrightarrow^{\ast} B \) are satisfied!!
The Invariant Paradox

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rl < s(R), W > => < R, W > .
endm

Q: How can we express its mutual exclusion invariant as a reachability formula \( A \longrightarrow^\circ B \)?

A: Since: (i) \( A \longrightarrow^\circ B \) just means \( A \rightarrow (\Box \text{enabled}) \lor \Diamond B \), and (ii) READERS–WRITERS is a never terminating rewrite theory, all formulas \( A \longrightarrow^\circ B \) are satisfied!! So we cannot!!
Consider the readers and writers example (Lecture 18):

mod READERS-WRITERS is
  protecting NAT .
sort State .
op <_,_> : Nat Nat -> State [ctor] . --- readers/writers
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rl < R, s(W) > => < R, W > .
rl < R, 0 > => < s(R), 0 > .
rl < s(R), W > => < R, W > .
endm

Q: How can we express its *mutual exclusion* invariant as a reachability formula \( A \rightarrow^\star B \)?

A: Since: (i) \( A \rightarrow^\star B \) just means \( A \rightarrow (□enabled) \lor \diamond B \), and (ii) READERS-WRITERS is a *never terminating* rewrite theory, all formulas \( A \rightarrow^\star B \) are satisfied!! So we cannot!! (Paradox!!).
Solving the Invariant Paradox

Let us add a *stopwatch* to READERS–WRITERS as follows:
Solving the Invariant Paradox

Let us add a *stopwatch* to READERS–WRITERS as follows:

mod READERS–WRITERS–stop is
  protecting NAT .
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  vars R W : Nat .
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  rl < R, s(W) > => < R, W > .
  rl < R, 0 > => < s(R), 0 > .
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  rl < s(R), W > => < R, W > .
  endm

The rule < R, W > => [R,W] can now *stop* any state and make it terminating.
```
Solving the Invariant Paradox

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\[
\text{rl < s(R), W > => < R, W > .}
\]

\[
\text{rl < R, W > => [R,W] .}
\]

\[
\text{endm}
\]

The rule \( < R, W > \Rightarrow [R,W] \) can now *stop* any state and make it terminating. For any pattern predicate \( B = \langle u, v \rangle \mid \varphi \) let \([B]\) denote the pattern predicate \([B] = [u, v] \mid \varphi\).
Let us add a *stopwatch* to READERS–WRITERS as follows:

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  rl < s(R), W > => < R, W > .
endm
```

The rule $< R, W > => [R,W]$ can now *stop* any state and make it terminating. For any pattern predicate $B = \langle u, v \rangle | \varphi$ let $[B]$ denote the pattern predicate $[B] = [u, v] | \varphi$.

**Fact.** $B$ is an *invariant* from states $S_0$ in READERS–WRITERS iff
Let us add a *stopwatch* to READERS–WRITERS as follows:

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  rl < R, 0 > => < s(R), 0 > .
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endm
```

The rule `< R, W > => [R,W]` can now *stop* any state and make it terminating. For any pattern predicate $B = \langle u, v \rangle | \varphi$ let $[B]$ denote the pattern predicate $[B] = [u, v] | \varphi$.

**Fact.** $B$ is an *invariant* from states $S_0$ in READERS–WRITERS iff $S_0 \xrightarrow{\otimes} [B]$ holds in in READERS–WRITERS–stop.
Suppose \( \mathcal{R} \) is *never terminating* (has no terminating states),
Solving the Invariant Paradox (General Case)

Suppose $\mathcal{R}$ is never terminating (has no terminating states), $State$ has a single constructor $\langle \_ , \_ , \_ \rangle : s_1 \ldots s_n \rightarrow State$, 
Solving the Invariant Paradox (General Case)

Suppose $\mathcal{R}$ is never terminating (has no terminating states), $State$ has a single constructor $\langle \_, \ldots, \_ \rangle : s_1 \ldots s_n \rightarrow State$, and all rules are between terms of sort $State$. 

Example. Mutual exclusion from $\langle 0, 0 \rangle$ in READERS-WRITERS is the predicate: $Mutex = \langle R, W \rangle | W = 0 \lor (W = 1 \land R = 0)$. We can prove it by showing:

(i) $\langle 0, 0 \rangle \in Mutex$ (easy), and

(ii) $Mutex \rightarrow \sqcup Mutex_{\sigma}$ in READERS-WRITERS-stop.
Suppose $\mathcal{R}$ is *never terminating* (has no terminating states), $State$ has a single constructor $\langle _, \ldots, _ \rangle : s_1 \ldots s_n \rightarrow State$, and all rules are between terms of sort $State$. Call $\mathcal{R}_{stop}$ the rewrite theory extending $\mathcal{R}$ by adding:

\begin{itemize}
  \item (i) $\langle _, \ldots, _ \rangle : s_1 \ldots s_n \rightarrow State$
  \item (ii) a *stop rule* $\langle x_1, \ldots, x_n \rangle \rightarrow \langle x_1, \ldots, x_n \rangle$
\end{itemize}

Then:

**Theorem** $B$ is an invariant for $\mathcal{R}$ from initial states $S_0$ iff $S_0 \xrightarrow{\ast} \ast B$ holds in $\mathcal{R}_{stop}$.

**Corollary** If $J_{S_0 K} \subseteq J_{B K}$ and $B \xrightarrow{\ast} \ast B_{\sigma}$ holds in $\mathcal{R}_{stop}$, then $B$ is an invariant for $\mathcal{R}$ from initial states $S_0$.

**Example.** Mutual exclusion from $\langle 0, 0 \rangle$ in READERS-WRITERS is the predicate:

$Mutex = \langle R, W \rangle | W = 0 \lor (W = 1 \land R = 0)$.

We can prove it by showing:

\begin{itemize}
  \item (i) $\langle 0, 0 \rangle \in Mutex$ (easy), and
  \item (ii) $Mutex \xrightarrow{\ast} \ast Mutex_{\sigma}$ in $\text{READERS-WRITERS-stop}$.
\end{itemize}
Suppose $\mathcal{R}$ is \textit{never terminating} (has no terminating states), $State$ has a single constructor $\langle _, \ldots, _ \rangle : s_1 \ldots s_n \rightarrow State$, and all rules are between terms of sort $State$. Call $\mathcal{R}_{stop}$ the rewrite theory extending $\mathcal{R}$ by adding: (i) $\langle _, \ldots, _ \rangle : s_1 \ldots s_n \rightarrow State$, and
Solving the Invariant Paradox (General Case)

Suppose $\mathcal{R}$ is \textit{never terminating} (has no terminating states), $State$ has a single constructor $\langle \_, \ldots, \_ \rangle : s_1 \ldots s_n \rightarrow State$, and all rules are between terms of sort $State$. Call $\mathcal{R}_{stop}$ the rewrite theory extending $\mathcal{R}$ by adding: (i) $\langle \_, \ldots, \_ \rangle : s_1 \ldots s_n \rightarrow State$, and (ii) a \textit{stop rule} $\langle x_1, \ldots, x_n \rangle \rightarrow [x_1, \ldots, x_n]$. 
Solving the Invariant Paradox (General Case)

Suppose $\mathcal{R}$ is *never terminating* (has no terminating states), $State$ has a single constructor $\langle _, \ldots, _ \rangle : s_1 \ldots s_n \rightarrow State$, and all rules are between terms of sort $State$. Call $\mathcal{R}_{\text{stop}}$ the rewrite theory extending $\mathcal{R}$ by adding: (i) $[_, \ldots, _] : s_1 \ldots s_n \rightarrow State$, and (ii) a *stop rule* $\langle x_1, \ldots, x_n \rangle \rightarrow [x_1, \ldots, x_n]$. Then:

Theorem $\mathcal{B}$ is an invariant for $\mathcal{R}$ from initial states $S_0$ iff $S_0 \rightarrow_\ast \ast \{ \mathcal{B} \}$ holds in $\mathcal{R}_{\text{stop}}$.

Corollary If $J S_0 K \subseteq J \mathcal{B} K$ and $\mathcal{B} \rightarrow_\ast \ast \{ \mathcal{B} \sigma \}$ (σ variable renaming) holds in $\mathcal{R}_{\text{stop}}$, then $\mathcal{B}$ is an invariant for $\mathcal{R}$ from initial states $S_0$.

Example. Mutual exclusion from $\langle 0, 0 \rangle$ in READERS-WRITERS is the predicate: $\text{Mutex} = \langle R, W \rangle | W = 0 \lor (W = 1 \land R = 0)$. We can prove it by showing: (i) $\langle 0, 0 \rangle \in \text{Mutex}$ (easy), and (ii) $\text{Mutex} \rightarrow_\ast \ast \{ \text{Mutex} \sigma \}$ in READERS-WRITERS-stop.
Solving the Invariant Paradox (General Case)

Suppose $\mathcal{R}$ is *never terminating* (has no terminating states), $\text{State}$ has a single constructor $\langle \_, \ldots, \_ \rangle : s_1 \ldots s_n \rightarrow \text{State}$, and all rules are between terms of sort $\text{State}$. Call $\mathcal{R}_{\text{stop}}$ the rewrite theory extending $\mathcal{R}$ by adding: (i) $\langle \_, \ldots, \_ \rangle : s_1 \ldots s_n \rightarrow \text{State}$, and (ii) a *stop rule* $\langle x_1, \ldots, x_n \rangle \rightarrow [x_1, \ldots, x_n]$. Then:

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$B$ is an *invariant* for $\mathcal{R}$ from initial states $S_0$ iff $S_0 \rightarrow^\ast [B]$ holds in $\mathcal{R}_{\text{stop}}$. 

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If $[S_0] \subseteq [B]$ and $B \longrightarrow^\ast [B\sigma]$ ($\sigma$ variable renaming) holds in $\mathcal{R}_{\text{stop}}$, then $B$ is an \textit{invariant} for $\mathcal{R}$ from initial states $S_0$. 

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\begin{align*}
\text{Mutex} &= \langle R, W \rangle | W = 0 \lor (W = 1 \land R = 0) \\
\text{We can prove it by showing:} (i) \langle 0, 0 \rangle &\in \text{Mutex} \text{(easy)}, \text{and} (ii) \text{Mutex} \longrightarrow^\ast [\text{Mutex}\sigma] \text{in} \ \text{READERS-WRITERS-stop}.
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Solving the Invariant Paradox (General Case)

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**Theorem**

$B$ is an invariant for $\mathcal{R}$ from initial states $S_0$ iff $S_0 \rightarrow \star [B]$ holds in $\mathcal{R}_{\text{stop}}$.

**Corollary**

If $[S_0] \subseteq [B]$ and $B \rightarrow \star [B\sigma]$ (\(\sigma\) variable renaming) holds in $\mathcal{R}_{\text{stop}}$, then $B$ is an invariant for $\mathcal{R}$ from initial states $S_0$.

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*B is an invariant* for $\mathcal{R}$ from initial sates $S_0$ iff $S_0 \rightarrow^\ast \llbracket B \rrbracket$ holds in $\mathcal{R}_{\text{stop}}$.

**Corollary**

*If* $\llbracket S_0 \rrbracket \subseteq \llbracket B \rrbracket$ *and* $B \rightarrow^\ast \llbracket B\sigma \rrbracket$ (*$\sigma$ variable renaming*) *holds in* $\mathcal{R}_{\text{stop}}$, *then* $B$ *is an invariant* for $\mathcal{R}$ *from initial sates* $S_0$.

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If \( \llbracket S_0 \rrbracket \subseteq \llbracket B \rrbracket \) and \( B \rightarrow^\ast [B\sigma] \) (\( \sigma \) variable renaming) holds in \( \mathcal{R}_{\text{stop}} \), then \( B \) is an \textit{invariant} for \( \mathcal{R} \) from initial states \( S_0 \).

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\( B \) is an *invariant* for \( \mathcal{R} \) from initial states \( S_0 \) iff \( S_0 \rightarrow^* [B] \) holds in \( \mathcal{R}_{\text{stop}} \).

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*If* \( [S_0] \subseteq [B] \) *and* \( B \rightarrow^* [B\sigma] \) (*\( \sigma \) variable renaming) *holds in* \( \mathcal{R}_{\text{stop}} \), *then* \( B \) *is an invariant* for \( \mathcal{R} \) *from initial states* \( S_0 \).

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Q: Then given RWL theory $\mathcal{R}$, how do we prove $A \rightarrow^\oplus B$?

A: Perhaps surprisingly, two proof rules are enough:

1. A rule that traces rewrite steps of symbolic states in $\mathcal{R}$.
2. A rule that performs inductive reasoning in $\mathcal{R}$.

We call these two rules Step+Subsumption and Axiom, respectively.

The key ideas are:

(i) to prove $A \rightarrow^\oplus B$ we may need some auxiliary lemmas. Call $C$ the formula $A \rightarrow^\oplus B$ plus these lemmas;

(ii) we start with labeled sequents of the form $\left[\emptyset, C\right] \vdash T\u | \varphi \rightarrow^\oplus \bigvee_i \nu_i | \psi_i$ for all formulas in $C$;

(iii) the first component ($\emptyset$) are the formulas we can assume as axioms (none);

(iv) the second ($C$) are the formulas we need to prove and cannot yet assume;

(v) the Step+Subsumption rule allows us to inductively assume $C$ after a rewrite step with rules $\mathcal{R} = \{ l_j \rightarrow r_j | \varphi_j \}$.
Q: Then given RWL theory $\mathcal{R}$, how do we prove $A \rightarrow^\otimes B$?

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Reachability Logic
Proof Rules

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Reachability Logic
Proof Rules

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Reachability Logic

Proof Rules

\[ \bigwedge_{(j, \alpha) \in \text{UNIFY}(u | \varphi', R)} \quad [A \cup C, \emptyset] \vdash_T (r_j | \varphi' \land \phi_j)\alpha \longrightarrow^* \bigvee_i (v_i | \psi_i)\alpha \]

\[ [A, C] \vdash_T u | \varphi \longrightarrow^* \bigvee_i v_i | \psi_i \]

\[ \bigwedge_j [\{u' | \varphi' \longrightarrow^* \bigvee v'_j | \psi'_j\} \cup A, \emptyset] \vdash_T v'_j\alpha | \varphi \land \psi'_j\alpha \longrightarrow^* \bigvee_i v_i | \psi_i \]

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The Step+Subsumption and Axiom Rules
Q: So what work has been done already?

A: A substantial RL framework is already in place with:
- full semantics for RL developed in terms of RWL
- soundness proof for proof system and semantics
- Maude tool semi-automating the proof system
- a collection of case studies.

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