Program Verification: Lecture 19

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More on Reachability Homomorphisms

Recall that given two \((\Sigma, \phi)\)-reachability models \(A_\rightarrow = (A, \rightarrow_A)\), and \(B_\rightarrow = (B, \rightarrow_B)\), a \((\Sigma, \phi)\)-reachability homomorphism \(h : A_\rightarrow \rightarrow B_\rightarrow\) is a \(\Sigma\)-homomorphism \(h : A \rightarrow B\) such that “preserves reachability,” that is, for each \(k \in K\), \(a \rightarrow_{A,k} a'\) implies \(h_k(a) \rightarrow_{B,k} h_k(a')\).

We call a reachability homomorphism \(h : A_\rightarrow \rightarrow B_\rightarrow\) a (stuttering) bisimulation if, in addition, for each \(k \in K\), and each \(a \in A_k\), \(h_k(a) \rightarrow_{B,k} b'\) implies that there exists \(a' \in A_k\) such that: (i) \(h_k(a') = b'\), and (ii) \(a \rightarrow_{A,k} a'\). That is, a bisimulation preserves reachability “in both directions.”
More on Reachability Homomorphisms (II)

Recall that, given any \((\Sigma, \phi)\)-reachability model \(\mathcal{A} \to = (\mathcal{A}, \to_A)\) a kind \(k\) and an element \(a \in A_k\), we defined \(\text{Reach}_{\mathcal{A} \to} (a) = \{x \in A_k \mid a \to_A x\}\).

Recall also that, by definition, if \(I\) is a Boolean-valued predicate in \(\Sigma\) defining and invariant, and \(a \in A\), we have

\[\mathcal{A} \to, a \models \Box I\]

iff \(\forall a' \in \text{Reach}_{\mathcal{A} \to} (a) \ I_A(a') = \text{true}_A\)
Reachability homomorphisms allow us to shift our ground in the verification process. Using a reachability homomorphism $h : \mathcal{A}_{\to} \rightarrow \mathcal{B}_{\to}$ we can reduce proving an invariant for $\mathcal{A}_{\to}$ (which may be infinite-state or too big) to proving an invariant for $\mathcal{B}_{\to}$ (which may be finite-state or smaller).

**Theorem.** Suppose now that we have a $(\Sigma, \phi)$-reachability homomorphism $h : \mathcal{A}_{\to} \rightarrow \mathcal{B}_{\to}$ and that both $\mathcal{A}_{\to} = (\mathcal{A}, \rightarrow_{\mathcal{A}})$, and $\mathcal{B}_{\to} = (\mathcal{B}, \rightarrow_{\mathcal{B}})$ protect $\text{Bool}$. Then, for any Boolean predicate in $\Sigma$ we have the implication

$$\mathcal{B}_{\to}, h(a) \models \Box I \quad \Rightarrow \quad \mathcal{A}_{\to}, a \models \Box I$$

Furthermore, if $h$ is a bisimulation this is an equivalence.
Proving Invariants with Reachability Homomorphisms (II)

Proof: We can prove the \((\Rightarrow)\) implication by contradiction. Suppose \(B \rightarrow, h(a) \models \Box I\) holds but \(A \rightarrow, a \not\models \Box I\). We then have an \(a' \in A\) with \(a \rightarrow_A a'\) and \(I_A(a') = \text{false}_A\). But by \(h\) reachability homomorphism we must have \(h(a) \rightarrow_B h(a')\) and also \(I_B(h(a')) = h(I_A(a')) = \text{false}_A\), contradicting \(B \rightarrow, h(a) \models \Box I\).

If \(h\) is a bisimulation, we can prove the \((\Leftarrow)\) implication by contradiction. Suppose \(A \rightarrow, a \models \Box I\) and \(B \rightarrow, h(a) \not\models \Box I\). We then have some \(b \in B\) with \(h(a) \rightarrow_A b\) and \(I_B(b) = \text{false}_B\). But by \(h\) bisimulation we have \(a \rightarrow_A a'\) with \(h(a') = b\), and by \(A \rightarrow, a \models \Box I\) we have \(I_A(a') = \text{true}_A\), which forces \(I_B(b) = \text{true}_B\), contradicting \(I_B(b) = \text{false}_B\). q.e.d.
Equational Abstractions

A very simple method to exploit the above theorem is to use an **equational abstraction**. The idea is enormously simple. Suppose that our system has been specified by means of a rewrite theory $\mathcal{R} = (\Sigma, E, \phi, R)$ which we assume satisfies all the executability conditions and protects the sort $\textbf{Bool}$. We can then **add new equations**, say $G$ to $\mathcal{R}$ to obtain a rewrite theory $\mathcal{R}/G = (\Sigma, E \cup G, \phi, R)$. Now consider the initial model $\mathcal{T}_{\mathcal{R}/G}$. By construction $\mathcal{T}_{\mathcal{R}/G}$ satisfies $E \cup G$ (in particular $E$), and $R$. Therefore, by the initiality theorem for $\mathcal{T}_\mathcal{R}$ we have a unique reachability homomorphism

$$-\mathcal{T}_{\mathcal{R}/G} : \mathcal{T}_\mathcal{R} \rightarrow \mathcal{T}_{\mathcal{R}/G}$$

mapping each $[t]_E$ to $[t]_{E \cup G}$. We call $\mathcal{T}_{\mathcal{R}/G}$ the **equational abstraction** by equations $G$ of $\mathcal{T}_\mathcal{R}$. 
Suppose that $\mathcal{R}/G$ poects $\text{Bool}$. Then, for any invariant $I$ of interest, our previous theorem immediately applies to give as an implication:

$$\mathcal{T}_{\mathcal{R}/G}, [t]_{E\cup G} \models \square I \quad \Rightarrow \quad \mathcal{T}_{\mathcal{R}}, [t]_E \models \square I$$

Therefore, we can use the equational abstraction $\mathcal{T}_{\mathcal{R}/G}$, which typically is much smaller and can even be finitely reachable when $\mathcal{T}_{\mathcal{R}}$ is infinitely reachable, to verify the invariant $\mathcal{T}_{\mathcal{R}}, [t]_E \models \square I$. But to do this in Maude we need, besides checking the requirement that $\mathcal{R}/G$ poects $\text{Bool}$, to also check that $\mathcal{R}/G$ satisfies the usual executability requirements, namely, that it is ground confluent, sort decreasing, and terminating, and also that it is ground coherent.
We can illustrate the power of equational abstractions with our readers and writers example, for which we already performed bounded model checking of invariants up to depth $10^6$ in Lecture 22. Since the Maude Church-Rosser Checker and Coherence Checker tools do not currently allow built-in submodules like NAT or BOOL, we will consider the following slight variants of our original specifications that do not use any built-ins.

\begin{verbatim}
mod R&W is
  sort Nat Config .
  op <_,_> : Nat Nat -> Config [ctor] . --- readers/writers
  op 0 : -> Nat [ctor] .
  op s : Nat -> Nat [ctor] .
  vars R W : Nat .
\end{verbatim}
\[ \text{rl} < 0, 0 > \Rightarrow < 0, s(0) > . \]
\[ \text{rl} < R, s(W) > \Rightarrow < R, W > . \]
\[ \text{rl} < R, 0 > \Rightarrow < s(R), 0 > . \]
\[ \text{rl} < s(R), W > \Rightarrow < R, W > . \]

```plaintext
endm
```

```plaintext
mod R&W-PREDS is
    protecting R&W.
    sort NewBool.
    ops tt ff : -> NewBool [ctor].
    ops mutex one-writer : Config -> NewBool [frozen].
    eq mutex(< s(N:Nat),s(M:Nat) >) = ff .
    eq mutex(< 0,N:Nat >) = tt .
    eq mutex(< N:Nat,0 >) = tt .
    eq one-writer(< N:Nat,s(s(M:Nat)) >) = ff .
    eq one-writer(< N:Nat,0 >) = tt .
    eq one-writer(< N:Nat,s(0) >) = tt .
endm
```
We can then drastically collapse the set of states by defining the following equational abstraction, which we can immediately use to verify our two invariants:

mod R&W-ABS is
  including R&W-PREDs .
  eq < s(s(N:Nat)),0 > = < s(0),0 > .
endm

=================================================================
search in R&W-ABS : < 0,0 > =>* C:Config such that
  mutex(C:Config) = ff .
No solution.
=================================================================
search in R&W-ABS : < 0,0 > =>* C:Config such that
  one-writer(C:Config) = ff .
No solution.
A Readers&Writers Example (III)

Since Maude computes really in the canonical reachability model $C_\mathcal{R}$ of the given rewrite theory $\mathcal{R}$ which is only isomorphic to the initial model $T_\mathcal{R}$ under the required executability assumptions, in reality we are not yet finished verifying that our original readers and writers system satisfies the two invariants using the above equational abstraction. We furthermore need to show that:

1. both R&W-PREDS and R&W-ABS protect NewBool;

2. R&W-ABS is ground confluent, sort-decreasing, and terminating; and

3. R&W-ABS is ground coherent.
Note that we can easily show (1): that both R&W-PREDS and R&W-ABS protect NewBool, by showing that: (i) both are ground confluent, sort-decreasing, and terminating; and (ii) both are sufficiently complete. Indeed, by (i), the canonical term algebra by the equations is isomorphic to the initial algebra; and by (ii), the only canonical terms of sort NewBool must be the constructors tt and ff, which are both in canonical form, and therefore different. Note also that (i) above shows (2) as well.
We can check local confluence and sort-decreasingness of\nR&W-PREDs and R&W-ABS with the CRC tool:

Maude> (check Church-Rosser R&W-PREDs .)
Checking solution:
  All critical pairs have been joined. The specification is
    locally-confluent.
  The specification is sort-decreasing.

Maude> (check Church-Rosser R&W-ABS .)
Checking solution:
  All critical pairs have been joined. The specification is
    locally-confluent.
  The specification is sort-decreasing.
Since the termination of the equations in R&W-ABS implies that of the (fewer) equations in R&W-PREDs, it is enough to check the first module. After extracting a functional module with the equations from R&W-ABS and with a slight change of syntax and using AProVe through the MTT tool (with no sort information) we get:

(fmod RWABS is

sort Nat Config .


op s : Nat -> Nat [ctor] .

vars R W : Nat .

sort NewBool .)
op tt : \rightarrow NewBool \ [ctor] .
op ff : \rightarrow NewBool \ [ctor] .
op mutex : Config \rightarrow NewBool .
op one-writer : Config \rightarrow NewBool .

vars M N : Nat .

eq \ mutex(cg(N, s(M))) = ff .
\eq \ mutex(cg(zero, N)) = tt .
\eq \ mutex(cg(N, zero)) = tt .
\eq \ one-writer(cg(N, s(s(M)))) = ff .
\eq \ one-writer(cg(N, zero)) = tt .
\eq \ one-writer(cg(N, s(zero))) = tt .
\eq \ cfg(s(s(N)), zero) = cfg(s(zero),zero) .

endfm)

*** AProVe output

Termination of R to be shown.
Removing the following rules from \( R \) which fulfill a polynomial ordering:

\[
\begin{align*}
\text{and}(tt, X) &\rightarrow X \\
\text{mutex}(\text{cfg}(s(N), s(M))) &\rightarrow ff \\
\text{one} - \text{writer}(\text{cfg}(N, s(\text{zero}))) &\rightarrow tt \\
\text{one} - \text{writer}(\text{cfg}(N, s(s(M)))) &\rightarrow ff
\end{align*}
\]

where the Polynomial interpretation:

\[
\begin{align*}
\text{POL}(\text{and}(x_{o1}, x_{o2})) &= x_{o1} + x_{o2} \\
\text{POL}(ff) &= 0 \\
\text{POL}(\text{mutex}(x_{o1})) &= x_{o1} \\
\text{POL}(tt) &= 1 \\
\text{POL}(s(x_{o1})) &= 1 + x_{o1}
\end{align*}
\]
POL(cfg(xo1, xo2)) = xo1 + xo2
POL(xo1 - xo2) = xo1 + xo2
POL(one) = 0
POL(zero) = 1
POL(writer(xo1)) = xo1

was used.

Not all Rules of R can be deleted, so we still have to regard a part of R.

\[ R \rightarrow RRRPolo \]
\[ \rightarrow TRS2 \]
\[ \rightarrow \text{Removing Redundant Rules} \]

Removing the following rules from R which fulfill a polynomial ordering:
cfg(s(s(N)), zero) -> cfg(s(zero), zero)

where the Polynomial interpretation:
POL(tt) = 0
POL(mutex(xo1)) = xo1
POL(s(xo1)) = 1 + xo1
POL(cfg(xo1, xo2)) = xo1 + xo2
POL(xo1 - xo2) = xo1 + xo2
POL(one) = 0
POL(zero) = 0
POL(writer(xo1)) = xo1
was used.

Not all Rules of R can be deleted, so we still have to regard a part of R.

R ->RRRPolo
Removing the following rules from R which fulfill a polynomial ordering:

one - writer(cfg(N, zero)) -> tt

where the Polynomial interpretation:
POL(mutex(xo1)) = xo1
POL(tt) = 0
POL(cfg(xo1, xo2)) = xo1 + xo2
POL(xo1 - xo2) = xo1 + xo2
POL(one) = 1
POL(zero) = 0
POL(writer(xo1)) = xo1
was used.

Not all Rules of R can be deleted, so we still have to regard a part of R.

\[
R \rightarrow \text{RRRPolo} \\
\rightarrow \text{TRS2} \\
\rightarrow \text{RRRPolo} \\
\rightarrow \text{TRS3} \\
\rightarrow \text{RRRPolo} \\
\ldots \\
\rightarrow \text{TRS4} \\
\rightarrow \text{Removing Redundant Rules}
\]
Removing the following rules from R which fulfill a polynomial ordering:

\[
\text{mutex}(\text{cfg}(\text{zero}, N)) \rightarrow \text{tt} \\
\text{mutex}(\text{cfg}(N, \text{zero})) \rightarrow \text{tt}
\]

where the Polynomial interpretation:

\[
\text{POL}(\text{mutex}(x_0)) = x_0 \\
\text{POL}(\text{tt}) = 0 \\
\text{POL}(\text{cfg}(x_0, x_02)) = 1 + x_0 + x_02 \\
\text{POL}(\text{zero}) = 0
\]

was used.

All Rules of R can be deleted.

\[
R \rightarrow \text{RRRPolo}
\]
The TRS is overlay and locally confluent (all critical pairs are trivially joinable).
->TRS3
->RRRPolo
...

->TRS6
->Dependency Pair Analysis

R contains no Dependency Pairs and therefore no SCCs.

Termination of R successfully shown.

Duration:
0:00 minutes
Next we can use the SCC Tool to check the sufficient completeness of \( R&W\text{-PRED}S \) and \( R&W\text{-ABS} \) which, together with the confluence and termination checks already performed ensures that \( R&W\text{-PRED}S \) and \( R&W\text{-ABS} \) protect BOOL.

```
Maude> load scc
Maude> (scc R&W-PREDs .)
Checking sufficient completeness of R&W-PREDs ...
Success: R&W-PREDs is sufficiently complete under the assumption that it is weakly-normalizing, confluent, and sort-decreasing.
Maude> (scc R&W-ABS .)
Checking sufficient completeness of R&W-ABS ...
Success: R&W-ABS is sufficiently complete under the assumption that it is weakly-normalizing, confluent, and sort-decreasing.
```
The only remaining check is the ground coherence of R&W-ABS. Using the ChC tool get get:

Maude> (check coherence R&W-ABS .)

   The following critical pairs cannot be rewritten:
   cp < s(0),0 >
   => < s(N*@:Nat),0 > .

Gound coherence requires that all ground instances of such a pair can indeed be rewritten in one step by the rules. We can reason by cases and consider the canonical forms by our equation of the following instances of the righthand side:

- \( \text{can}(< s(0),0 >) = < s(0),0 > \)
- \( \text{can}(< s(s(N:\text{Nat})),0 >) = < s(0),0 > \)
So, all boils down to checking whether we can rewrite the term \(< s(0), 0 >\) to \textit{itself} in one step with the rules of the module, up to canonical form. This is indeed the case, since using rule

\[
rl < R, 0 > \Rightarrow < s(R), 0 > .
\]

we can rewrite \(< s(0), 0 >\) to \(< s(s(0)), 0 >\), whose canonical form is \(< s(0), 0 >\).