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Construction of the Initial Algebra $T_{\Sigma/E}$

$T_{\Sigma}$ is initial in the class $\text{Alg}_\Sigma$ of all $\Sigma$-algebras. To give an initial algebra semantics to Maude functional modules of the form $\text{fmod}(\Sigma, E) \text{endfm}$ we need an initial algebra in the class $\text{Alg}_{(\Sigma, E)}$ of all $(\Sigma, E)$-algebras, with $\Sigma$ sensible, kind complete, and with nonempty sorts.

We shall construct such an algebra, denoted $T_{\Sigma/E}$, and show that it is indeed initial in $\text{Alg}_{(\Sigma, E)}$, i.e., for any $(\Sigma, E)$-algebra $\mathcal{A}$ there is a unique $\Sigma$-homomorphism $\mathcal{A} : T_{\Sigma/E} \rightarrow \mathcal{A}$.

If the equations $E$ are sort-decreasing, ground confluent and operationally terminating will show that there is an isomorphism $T_{\Sigma/E} \cong C_{\Sigma/E}$, a very intuitive semantics.
We construct $\mathcal{T}_{\Sigma/E}$ out of the provability relation $(\Sigma, E) \vdash t = t'$; that is, out of the relation $t =_E t'$. But, by definition $t =_E t' \iff (\Sigma, \overrightarrow{E} \cup \overrightarrow{E}) \vdash t \rightarrow^* t'$. Therefore, $=_E$ besides being reflexive and transitive is symmetric, and therefore is an equivalence relation on terms. But since if $t =_E t'$, then there is a connected component $[s]$ such that $t, t' \in T_{\Sigma,[s]}$, in particular $=_{E}$ is also an equivalence relation on $T_{\Sigma,[s]}$. Therefore, we have a quotient set $T_{\Sigma/E,[s]} = T_{\Sigma,[s]}/=_E$.

We can then define the $S$-indexed family of sets $T_{\Sigma/E} = \{T_{\Sigma/E,s}\}_{s \in S}$, where, by definition,

$$T_{\Sigma/E,s} = \{[t] \in T_{\Sigma/E,[s]} \mid (\exists t') t' \in [t] \land t' \in T_{\Sigma,s}\},$$

where $[t]$, or $[t]_E$, abbreviate $[t] =_E$. 
To make \( T_{\Sigma/E} \) into a \( \Sigma \)-algebra \( T_{\Sigma/E} = (T_{\Sigma/E}, \mathcal{T}_{\Sigma/E}) \), interpret a constant \( a : \text{nil} \rightarrow s \) in \( \Sigma \) by its equivalence class \([a]\).

Similarly, given \( f : s_1 \ldots s_n \rightarrow s \) in \( \Sigma \), and given \([t_i] \in T_{\Sigma/E,s_i}, 1 \leq i \leq n\), define

\[
\left[ f^{s_1 \ldots s_n,s} \right]_{T_{\Sigma/E}} ([t_1], \ldots, [t_n]) = \left[ f(t'_1, \ldots, t'_n) \right],
\]

where \( t'_i \in [t_i] \land t'_i \in T_{\Sigma,s_i}, 1 \leq i \leq n \).

Checking that the above definition does not depend on either: (1) the choice of the \( t'_i \in [t_i] \), or (2) the choice of the subsort-overloaded operator \( f : s_1 \ldots s_n \rightarrow s \) in \( \Sigma \), so that it is well-defined and indeed defines an order-sorted \( \Sigma \)-algebra is left as an easy exercise.
**Theorem:** For $(\Sigma, E)$ with $\Sigma$ sensible, kind complete, and with nonempty sorts, $T_{\Sigma/E} \models E$. Furthermore, $T_{\Sigma/E}$ is initial in the class $\text{Alg}_{\Sigma, E}$. That is, for any $A \in \text{Alg}_{\Sigma, E}$ there is a unique $\Sigma$-homomorphism $E_A : T_{\Sigma/E} \rightarrow A$.  

**Proof:** We first need to show that $T_{\Sigma/E} \models E$, i.e., that $T_{\Sigma/E} \models t = t'$ for each $(t = t') \in E$. That is, for each assignment $a : X \rightarrow T_{\Sigma/E}$ we must show that $t a_{T_{\Sigma/E}} = t' a_{T_{\Sigma/E}}$.  

But the unique $\Sigma$-homomorphism $T_{\Sigma/E} : T_{\Sigma} \rightarrow T_{\Sigma/E}$ guaranteed by $T_{\Sigma}$ initial is just the passage to equivalence classes $t \mapsto [t]$ and therefore surjective.
Therefore, since by the Axiom of Choice any surjective function is a right inverse (STACS, Ch. 10, Thm. 9, pg. 80), we can always choose a substitution $\theta : X \rightarrow T_\Sigma$ such that $a = \theta; -T_\Sigma/E$. Therefore, by the Freeness Corollary we have $a_{T_\Sigma/E} = \theta; -T_\Sigma/E$ (see diagram next page).

Therefore, $t a_{T_\Sigma/E} = t' a_{T_\Sigma/E}$ is just the equality $[t\theta]_E = [t'\theta]_E$, which holds iff $t\theta =_E t'\theta$, which itself holds by $(t = t') \in E$ and the Lemma in the proof of the Soundness Theorem. Therefore, $T_\Sigma/E \models E$. 
Lifting of $\alpha$ to a Substitution $\theta$
Let us now show that for each $\mathcal{A} \in \text{Alg}_{(\Sigma, E)}$ there is a unique $\Sigma$-homomorphism $\overset{E}{\mathcal{A}} : \mathcal{T}_{\Sigma/E} \rightarrow \mathcal{A}$.

We first prove **uniqueness**. Suppose that we have two homomorphisms $h, h' : \mathcal{T}_{\Sigma/E} \rightarrow \mathcal{A}$. Then, composing with $\overset{E}{\mathcal{T}_{\Sigma/E}} : \mathcal{T}_{\Sigma} \rightarrow \mathcal{T}_{\Sigma/E}$ on the left we get, $\overset{E}{\mathcal{T}_{\Sigma/E}} ; h, \overset{E}{\mathcal{T}_{\Sigma/E}} ; h' : \mathcal{T}_{\Sigma} \rightarrow \mathcal{A}$, and by the initiality of $\mathcal{T}_{\Sigma}$ we must have, $\overset{E}{\mathcal{T}_{\Sigma/E}} ; h = \overset{E}{\mathcal{T}_{\Sigma/E}} ; h' = \overset{E}{\mathcal{A}}$. But recall that $\overset{E}{\mathcal{T}_{\Sigma/E}} : \mathcal{T}_{\Sigma} \rightarrow \mathcal{T}_{\Sigma/E}$ is **surjective**, and therefore (**Ex.9.1** epi), which forces $h = h'$, as desired.
To show existence of \( \varepsilon_{\mathcal{A}} : \mathcal{T}_{\Sigma/E} \rightarrow \mathcal{A} \), given \([t] \in T_{\Sigma/E,s}\), define \([t]_{\mathcal{A},s} = t'_{\mathcal{A},s}\), where \(t' \in [t] \land t' \in T_{\Sigma,s}\). Then show (exercise) that:

- \([t]_{\mathcal{A},s}\) is independent of the choice of \(t'\) because of the hypothesis \(\mathcal{A} \models E\) and the Soundness Theorem; and

- the family of functions \(\varepsilon_{\mathcal{A}} = \{\varepsilon_{\mathcal{A},s}\}_{s \in S}\) thus defined is indeed a \(\Sigma\)-homomorphism.

q.e.d.
As stated in pg. 2, the semantics of a Maude functional module \( \text{fmod}(\Sigma, E) \text{endfm} \) is an initial algebra semantics, given by \( T_{\Sigma/E} \). Let us call \( T_{\Sigma/E} \) the module's mathematical semantics. This semantics does not depend on any executability assumptions about \( \text{fmod}(\Sigma, E) \text{endfm} \): it can be defined for any equational theory \((\Sigma, E)\).

Call \( \text{fmod}(\Sigma, E) \text{endfm} \) admissible if the equations \( E \) are ground confluent, sort-decreasing, and terminating. Under these executability requirements we have another semantics for \( \text{fmod}(\Sigma, E) \text{endfm} \): the canonical term algebra \( C_{\Sigma/E} \) defined in Lecture 5. This is the most intuitive computational model for \( \text{fmod}(\Sigma, E) \text{endfm} \). Call it its operational semantics. But both semantics coincide!
The Canonical Term Algebra is Initial

**Theorem:** If the rules $\vec{E}$ are sort-decreasing, ground confluent and operationally terminating, then, $\mathcal{C}_{\Sigma/E}$ is isomorphic to $\mathcal{T}_{\Sigma/E}$ and is therefore initial in $\text{Alg}_{(\Sigma,E)}$.

**Proof:** An easy generalization of Ex.9.3 shows that if $\mathcal{I}$ is initial for a given class of algebras closed under isomorphisms and $\mathcal{J}$ is isomorphic to $\mathcal{I}$, then $\mathcal{J}$ is also initial for that class. Since (Ex.10.2) $\text{Alg}_{(\Sigma,E)}$ is closed under isomorphisms, we just have to show $\mathcal{T}_{\Sigma/E} \cong \mathcal{C}_{\Sigma/E}$.

Define $\text{can}_E = \{ \text{can}_{E,s} : T_{\Sigma/E,s} \rightarrow C_{\Sigma/E,s} \}_{s \in S}$ by, $\text{can}_{E,s}[t] = \text{can}_E(t)$. This is independent of the choice of $t$, since $t =_E t'$ iff $E \vdash t = t'$ iff (by $E$ confluent) $t \downarrow_E t'$, iff $\text{can}_E(t) = \text{can}_E(t')$. $\text{can}_{E,s}$ is surjective by construction and injective by these equivalences; therefore $\text{can}_E$ is bijective.
Let us see that $\text{can}_E : T_{\Sigma/E} \rightarrow C_{\Sigma/E}$ is a $\Sigma$-homomorphism. Preservation of constants is trivial. Let $f : s_1 \ldots s_n \rightarrow s$ in $\Sigma$, and $[t_i] \in T_{\Sigma/E,s_i}, 1 \leq i \leq n$. We must show,

$$\text{can}_{E,s}(f^{s_1 \ldots s_n,s}([t_1], \ldots, [t_n])) = f^{s_1 \ldots s_n,s}_{C_{\Sigma/E}}(\text{can}_E(t_1), \ldots, \text{can}_E(t_n)).$$

The key observation is that $\text{can}_E(t_i) \in T_{\Sigma,s_i}, 1 \leq i \leq n$. This is because:

- by definition of $[t_i]$ there must be a $t'_i \equiv_E t_i$ with $t'_i \in T_{\Sigma,s_i}, 1 \leq i \leq n$; and

- by the sort-decreasingness assumption for $E$, since $t'_i \longrightarrow^*_E \text{can}(t'_i) = \text{can}(t_i)$, if $t'_i \in T_{\Sigma,s_i}, 1 \leq i \leq n$, then $\text{can}_E(t_i) \in T_{\Sigma,s_i}, 1 \leq i \leq n$. 


Therefore, we have:

\[ \text{can}_{E,s}(f_{T_{\Sigma/E}}^{s_1\ldots s_n,s}([t_1], \ldots, [t_n])) = \text{by definition of } f_{T_{\Sigma/E}}^{s_1\ldots s_n,s} \]
\[ \text{can}_{E,s}([f_{\text{can}_E}(t_1), \ldots, f_{\text{can}_E}(t_n)]) = \text{by definition of } \text{can}_{E,s} \]
\[ \text{can}_E(f_{\text{can}_E}(t_1), \ldots, f_{\text{can}_E}(t_n)) = \text{by definition of } f_{C_{\Sigma/E}}^{s_1\ldots s_n,s} \]
\[ f_{C_{\Sigma/E}}^{s_1\ldots s_n,s}(\text{can}_E(t_1), \ldots, \text{can}_E(t_n)) \]

as desired.

All now reduces to proving the following easy lemma, which is left as an exercise:

**Lemma.** The bijective \( S \)-sorted map \( \text{can}_{E}^{-1} : C_{\Sigma/E} \rightarrow T_{\Sigma/E} \) is a \( \Sigma \)-homomorphism \( \text{can}_{E}^{-1} : C_{\Sigma/E} \rightarrow T_{\Sigma/E} \).

q.e.d
The canonical term algebra $\mathcal{C}_{\Sigma/E}$ is in some sense the most intuitive representation of the initial algebra from a computational point of view. Let us see in a simple example what the coincidence between mathematical and operational semantics means.

For example, the equations in the NATURAL module are ground confluent and terminating. Its canonical forms are the natural numbers in Peano notation. And its operations are the successor and addition functions.

Indeed, given two Peano natural numbers $n, m$ the general definition of $f^c_{\mathcal{C}_{\Sigma/E}}$ specializes for $f = +$ to the definition of addition, $n +_{\text{NATURAL}} m = \text{can}_{\text{NATURAL}}(n + m)$, so that $+_{\mathcal{C}_{\text{NATURAL}}}$ is the addition function.

\[ T_{\Sigma_{\text{NATURAL}}/E_{\text{NATURAL}}} = \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
ppss0 & s0 + 0 & ss0 + 0 & \cdots \\
0 + 0 & 0 + s0 & s0 + s0 & \cdots \\
ps0 & pss0 & psss0 & \cdots \\
0 & s0 & ss0 & \cdots \\
\end{array} \]

\[ C_{\Sigma_{\text{NATURAL}}/E_{\text{NATURAL}}} \]
More generally, we are interested in the agreement between the mathematical and operational semantics of an admissible Maude module of the form $\text{fmod}(\Sigma, E \cup B)\text{endfm}$, with $A$ a (possibly empty) set of associativity, commutativity, and identity axioms. The following, easy but nontrivial, generalization of the above theorem is left as an exercise.

**Theorem:** Let the equations $E$ in $(\Sigma, E \cup B)$ be sort-decreasing, ground confluent and weakly operationally terminating modulo $B$; and let $\Sigma$ be preregular modulo $B$. Then, $C_{\Sigma, E/B}$ is isomorphic to $T_{\Sigma/E \cup B}$ and is therefore initial in $\text{Alg}_{(\Sigma, E \cup B)}$. 
We are now ready to begin discussing program verification for deterministic declarative programs, and, more specifically, for Maude functional modules of the form \( \text{fmod}(\Sigma, E \cup B) \text{endfm} \), where we assume \( E \) ground confluent, sort-decreasing, and weakly operationally terminating modulo \( B \), and \( \Sigma \) preregular modulo \( B \). Their mathematical semantics is given by the initial algebra \( T_{\Sigma/E \cup B} \).

Their (concrete) operational semantics is given by equational simplification with \( E \) modulo \( B \). Both semantics coincide in the canonical term algebra, since we have the \( \Sigma \)-isomorphism,

\[
T_{\Sigma/E \cup B} \cong C_{\Sigma,E/B}.
\]
What are properties of a module $\text{fmod}(\Sigma, E \cup B)\text{endfm}$?

They are sentences $\varphi$, perhaps in equational logic, or, more generally, in first-order logic, in the language of a signature containing $\Sigma$.

When do we say that the above module satisfies property $\varphi$?

When we have,

$$\mathcal{T}_{\Sigma/E \cup B} \models \varphi.$$

How do we verify such properties?
Consider the module,

```plaintext
fmod NATURAL is
    sort Natural .
    op 0 : -> Natural [ctor] .
    op s : Natural -> Natural [ctor] .
    op _+_ : Natural Natural -> Natural .
    vars N M L : Natural .
    eq N + 0 = N .
    eq N + s(M) = s(N + M) .
endfm
```

A property $\varphi$ satisfied by this module is the **associativity** of addition, that is, the equation,

$$(\forall N, M, L) \ N + (M + L) = (N + M) + L.$$
Since the initial algebra $T_{\Sigma/E \cup B}$ associated to a module $\text{fmod}(\Sigma, E \cup B)$ satisfies the equations $E \cup B$, by the **Soundness Theorem** for equational deduction, whenever we can prove an equation $\varphi$ by $E \cup B \vdash \varphi$, we must have $T_{\Sigma/E \cup B} \models \varphi$, and therefore the module satisfies $\varphi$.

Therefore, equational deduction is always a **sound proof method** to verify properties of functional modules. However, it is quite limited, and generally insufficient for many properties.

In particular, for $\varphi$ the associativity of addition and $E$ the equations in NATURAL (in this case $A = \emptyset$) we cannot prove $E \vdash \varphi$. 


Need More than Equational Deduction (II)

This is easy to see, since associativity is not a property satisfied by all models of $E$. Consider, for example, the initial model obtained by adding a nonstandard number $a$,

```plaintext
fmod NON-STANDARD-NAT is
  sort Natural .
  ops 0 a : -> Natural [ctor] .
  op s : Natural -> Natural [ctor] .
  op _+_ : Natural Natural -> Natural .
  vars N M L : Natural .
  eq N + 0 = N .
  eq N + s(M) = s(N + M) .
endfm
```

This initial model satisfies $E$, but does not satisfy associativity, since $a + (a + a) \neq (a + a) + a$. In fact, no equations apply to either side. Therefore,

$E \not\vdash x + (y + z) = (x + y) + z$. 

Inductive Properties

The point is that associativity is an inductive property of natural number addition; that is, one satisfied by the initial model of $E$, but not in general by other models of $E$.

What we need are inductive proof methods based on a more powerful proof system $\vdash_{ind}$, satisfying the soundness requirement,

$$E \cup B \vdash_{ind} \phi \implies \mathcal{T}_{\Sigma/E \cup B} \models \phi.$$  

Also, it should prove all that equational deduction can prove and more. That is, for formulas $\varphi$ that are equations it should satisfy,

$$E \cup B \vdash \varphi \implies E \cup B \vdash_{ind} \varphi.$$
Inductive Properties (II)

Because of Gödel’s Incompleteness Theorem, in general we cannot hope to have completeness of inductive inference, that is, to have an equivalence

\[ E \cup B \vdash_{\text{ind}} \phi \iff \mathcal{T}_{\Sigma/E \cup B} \models \phi \]

although this may be possible for some very specific theories \((\Sigma, E)\) for which a complete proof system, or even an algorithm (a decision procedure), providing this equivalence exists.

The inductive inference system that we will justify and use generalizes the usual proofs by natural number induction. In fact, in our example of associativity of natural number addition it actually specializes to the usual proof method by natural number induction.
Sufficient Completeness is Crucial for Inductive Proofs

For the above signature $\Sigma$ and equations $E$, and $T_{\Sigma/E}$ the initial algebra, $T_{\Sigma/E} \not \models a + (a + a) = (a + a) + a$, since both terms are in canonical form and the equations are confluent. However, natural number induction on the declared constructors easily proves associativity of $\oplus$. Therefore, induction without sufficient completeness is unsound.
Exercises

• Consider the NAT-PREFIX specification of Lecture 2. Prove that the natural numbers $\mathbb{N}$, with zero, successor and the addition function are isomorphic to the initial algebra of that specification.

• Give your own algebraic specification of the Booleans in Maude (use a sort, say Truth, and constants $\texttt{tt}$, $\texttt{ff}$, to avoid any confusion with the built-in module BOOL in Maude) with disjunction, conjunction, and negation, and prove that the standard Booleans are isomorphic to the initial algebra of your specification.