Combinatory Logic and Curry Howard Isomorphism

Combinatory Logic   Schönfinkel (1924)

Syntax V is countable set of (program) variables

$$\mathcal{C} ::= \underset{\in V}{x} \mid K \mid S \mid (\mathcal{C}\ \mathcal{C})$$

constant fns

function application.

Notation $FGH \equiv ((F\ G)\ H)$

K : Function that takes 2 arguments and returns the first argument.

S : Function that takes 3 arguments and returns something.

Weak Reduction rule $\rightarrow_w \subseteq \mathcal{C} \times \mathcal{C}$.

"$F \rightarrow_w G$" : Program F reduces to program G in one step.

- $K\ F\ G \equiv ((K\ F)\ G)$

  $KFG \rightarrow_w F$

- $S\ F\ G\ H \equiv (((SF)\ G)\ H)$

  $SFGH \rightarrow_w FH(GH)$

- If $F \rightarrow_w F'$ then

  $FG \rightarrow_w F'G$ and

$FG \to_w F'G$ and

$$GF \to_w GF'$$

$\twoheadrightarrow_w$: Smallest reflexive transitive closure of $\to_w$

Examples

- Identity function : A function that takes an argument and the returns the same thing :

"$(Ix) = x$"

$$I \equiv ((SK)K)$$

$$(((SK)K)F) \to_w KF(KF)$$
$$\to_w F$$

- $\underline{\underline{B}}FGH = F(GH)$

$$B \equiv S(KS)K$$

Theorem Combinatory Logic is "Turing complete".

Example $SII(SII)$

"$SII(SII) \twoheadrightarrow_w SII(SII)$"

$$\underline{\underline{S}}II(SII) \to_w \underline{I(SII)}(\underline{I(SII)})$$
$$\to_w SII(SII)$$

Church-Roser Property If $F \twoheadrightarrow_w F_1$ and $F \twoheadrightarrow_w F_2$ then there is G. $F_1 \twoheadrightarrow_w G$ and $F_2 \twoheadrightarrow_w G$.

$$\begin{tikzcd}
 & F \arrow[dl, twoheadrightarrow] \arrow[dr, twoheadrightarrow] & \\
F_1 \arrow[dr, twoheadrightarrow] & & F_2 \arrow[dl, twoheadrightarrow] \\
 & \exists\, G &
\end{tikzcd}$$

Term in Normal Form F is a term in normal form if there is no G s.t $F \rightarrow_w G$.

Example S – normal form.

SK – normal form.

SKK – normal form.

Corollary If $F \twoheadrightarrow_w F_1$ and $F \twoheadrightarrow_w F_2$ and F_1 and F_2 are in normal form then $F_1 = F_2$.

Simply typed Combinatory Logic.

... set of type variables

Types U is a set of type variables

$$\tau ::= \underset{\in U}{\alpha} \mid (\tau \to \tau)$$

$\to$ is right associative

$a \to b \to c \equiv a \to (b \to c)$

Set of all Types to Π

$$C_\Pi ::= x \mid K_{\sigma,\tau} \mid S_{\sigma,\tau,\rho} \mid (C_\Pi \, C_\Pi)$$

Context is a set of the following form

$$\{x_1 : \alpha_1, \, x_2 : \alpha_2 \dots x_n : \alpha_n\}$$

where $x_1 \dots x_n \in V$, $\alpha_1 \dots \alpha_n \in \Pi$.

Typing Rules

$$C \vdash_c M : \alpha$$

In context C, program M has type α.

$$\frac{}{C, x:\alpha \vdash_c x : \alpha}$$

$$\frac{}{C \vdash_c K_{\sigma,\tau} : \sigma \to (\tau \to \sigma)}$$

$$\frac{}{C \vdash_c S_{\sigma,\tau,\rho} : (\sigma \to (\tau \to \rho)) \to ((\sigma \to \tau) \to (\sigma \to \rho))}$$

$$\frac{C \vdash_c F : \sigma \to \tau \qquad C \vdash_c G : \sigma}{C \vdash_c (F\ G) : \tau}.$$

$\underline{\bar{S} 1 1 (\bar{S} 1 1)}$ ← not a well typed program

Theorem

- If $C \vdash_c F : \sigma$ and $C \vdash_c F : \tau$ then $\sigma = \tau$.
- If $C \vdash_c F : \sigma$ and $F \to_\omega F'$ then $C \vdash_c F' : \sigma$.

Strong Normalization If $C \vdash_c F : \tau$ then there is no sequence $F \to_\omega F_1 \to_\omega F_2 \to \ldots$

$I_\sigma : \sigma \to \sigma$

$I_{\sigma \to \sigma} : (\sigma \to \sigma) \to (\sigma \to \sigma)$

$S_{\sigma,\tau,\rho} : (\sigma \to (\tau \to \rho)) \to ((\sigma \to \tau) \to (\sigma \to \rho))$

$S_{\sigma, \sigma \to \sigma, \rho} :$

Curry-Howard Isomorphism

$$\varphi ::= p \mid (\varphi \to \varphi)$$

$$\dfrac{}{\varphi \to (\psi \to \varphi)} \qquad \dfrac{}{\Gamma \vdash_c K_{\sigma,\tau} : \sigma \to (\tau \to \sigma)}$$

$$\dfrac{}{(\varphi \to (\psi \to \rho)) \to ((\varphi \to \psi) \to (\varphi \to \rho))} \qquad \dfrac{}{\Gamma \vdash_c S_{\sigma,\tau,\rho} : (\sigma \to (\tau \to \rho)) \to (\sigma \to \rho) \to \sigma \to \rho}$$

$$\dfrac{\varphi \quad \varphi \to \psi}{\psi} \qquad \dfrac{\Gamma \vdash F : \sigma \to \tau \quad \Gamma \vdash G : \sigma}{\Gamma \vdash (F\,G) : \tau}$$

Theorem $\Gamma \vdash \varphi$ then $\exists F$

$$\{x_i : \psi_i\}_{\psi_i \in \Gamma} \vdash_c F : \varphi$$

$C \vdash_c F : \varphi$ then $|C| \vdash \varphi$

$S_{p,p\to p,p} \vdash (p \to (p \to p) \to p) \to (p \to (p \to p)) \to (p \to p)$ (A)

$K_{p,p\to p} \vdash (p \to ((p \to p) \to p))$ (A)

$(SK) \vdash (p \to (p \to p)) \to (p \to p)$ (MP)

$K_{p,p} \vdash p \to (p \to p)$ (A)

$(SK)K \vdash p \to p$ (MP)

I

I : SKK