Courcelle's Theorem II

Theorem Given a graph $G = (V, E)$ of tree width w and an MSO sentence φ, $G \models \varphi$ can be determined in $O(f(|\varphi|, w) |V|)$

- Construct a Σ-labeled binary tree T and MSO sentence φ' over $\tau = (S_1, S_2, <, \{Q_a\}_{a \in \Sigma})$ such that $G \models \varphi$ iff $T \models \varphi'$.

Depends on w ~~and~~ ~~φ~~.

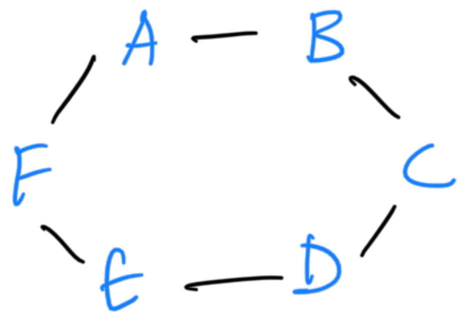
- Use Doner, Thatcher-Wright Theorem to construct a tree automaton $\mathcal{A}_{\varphi'}$ and run $\mathcal{A}_{\varphi'}$ on T.

Tree Decomposition of width w of G is a binary labeled tree $\tau(G) = (V_T, E_T, L_T)$ where $L_T : V_T \to V^{w+1}$ such that

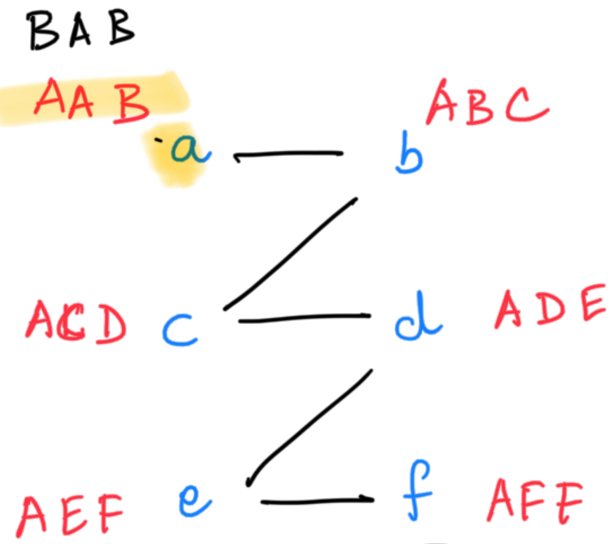
- (Node Coverage) For every $v \in V$, there is a $t \in V_T$ and $i \in \{0, 1 .. w\}$ such that
$L_T(t)[i] = v$
- (Edge Coverage) For every $(u, v) \in E$, there is a $t \in V_T$ and $i, j \in \{0, 1 ... w\}$ s.t.
$L_T(t)[i] = u$ and $L_T(t)[j] = v$

- (Coherence) For any $x, y \in V_T$ such that there is i and j with $L_T(x)[i] = L_T(y)[j]$ then for every z on path from x to y $\exists k$ s.t. $L_T(z)[k] = L_T(x)[i]$.

and $|V_T| = O(|V|)$



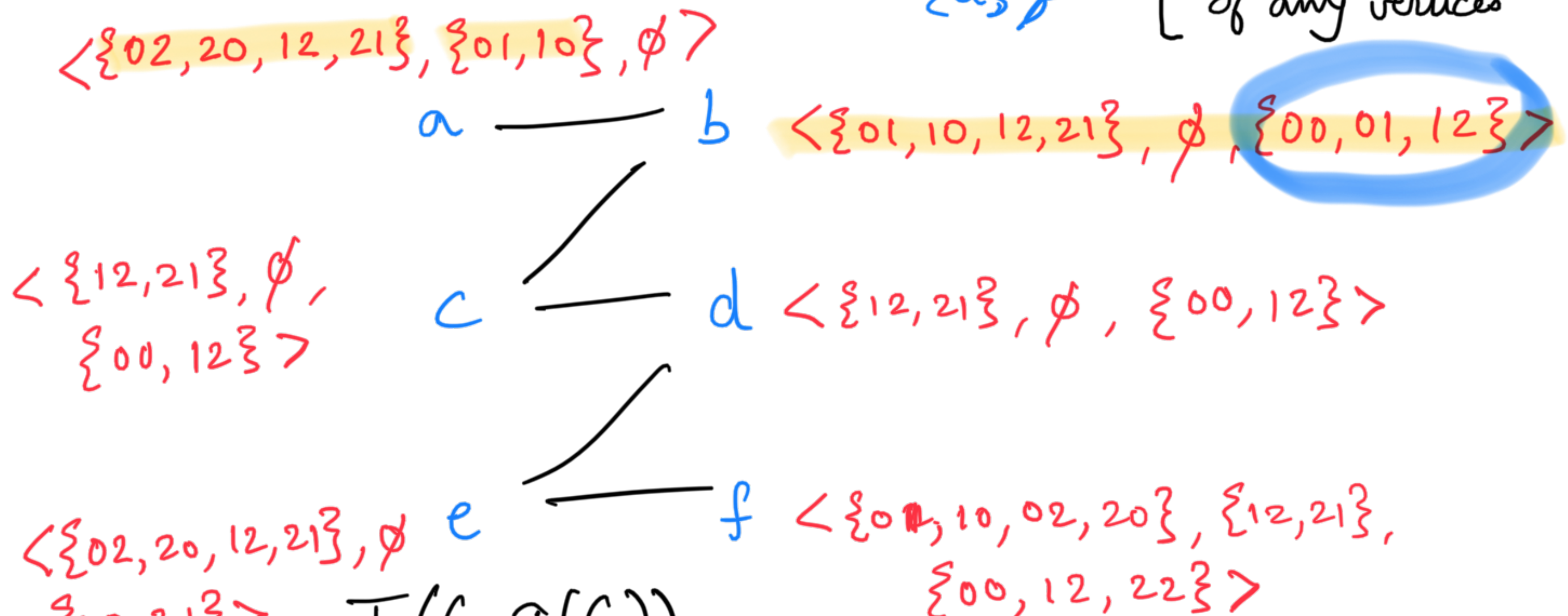
G



$\tau(G)$

Tree$(\{A,B\}) = \langle \{a,b,c,d,e,f\}, \{a,b\}, \{a\} \rangle$

$\langle \{b\}, \emptyset, \emptyset \rangle$ is not an encoding of any vertices



$T(G, \tau(G))$

Constructing tree with finite label set.

Given $G = (V, E)$ and tree decomposition $\tau(G) = (V_T, E_T, L_T)$ of width w. The tree $T(G, \tau(G)) = (V_T, E_T, L)$ s.t.

$l : V_T \to (\{0, 1, \ldots w\} \times \{0, 1, \ldots, w\})^3$

where $L(t) = (\lambda_1, \lambda_2, \lambda_3)$

- $\lambda_1 = \{ (i,j) \mid (L_T(t)[i], L_T(t)[j]) \in E \}$
- $\lambda_2 = \{ (i,j) \mid L_T(t)[i] = L_T(t)[j],\ i \neq j \}$
- $\lambda_3 = \{ (i,j) \mid L_T(t)[i] = L_T(x)[j]$ where x is the parent of $t \}$

$$\underbrace{G \models \varphi}_{\text{variables refer to vertices and sets of vertices of } G \text{, and edges of } G} \quad \text{iff} \quad \underbrace{T(G, \tau(G)) \models \varphi'}_{\tau = \{S_1, S_2, <, \{Q_a\}_{a\in\Sigma}\}}$$

variables refer to nodes / sets of nodes of $T(G, \tau(G))$, edges and labels of T.

$\forall x \exists y\ E(x,y) \longmapsto\ ?\ \forall \bar{X} \exists \bar{Y}\ \ \bar{X} \, \xi_T \, \bar{Y}$

correspond to vertices with an edge.

Encode sets of vertices of G as something in $T(G, \tau(G))$

$\mathrm{Tree}(S) = \langle U_0, U_1 \ldots U_w \rangle$

where $U_i \subseteq V_T$

$U_i = \{ t \mid L_T(t)[i] \in S \}$

Proposition $\langle U_0, U_1, U_2 \ldots U_w \rangle = \mathrm{Tree}(S)$ for some $S \subseteq V$ iff

(a) If $t \in U_i$ and $L(t) = \langle \lambda_1, \lambda_2, \lambda_3 \rangle$ and $(i,j) \in \lambda_2$ then $t \in U_j$.

(b) If $t \in U_i$, $L(t) = \langle \lambda_1, \lambda_2, \lambda_3 \rangle$ and x is parent of t, $(i,j) \in \lambda_3 \Rightarrow x \in U_j$.

Proposition $\langle U_0, U_1, \ldots U_w \rangle = \text{Tree}(\{u\})$ iff (a), (b) and

(c) If $t \in U_i \cap U_j$ then $(i,j) \in \lambda_2$ where $L(t) = \langle \lambda_1, \lambda_2, \lambda_3 \rangle$

(d) If $t \in U_i$ and $\text{parent}(t) \in U_j$ then $(i,j) \in \lambda_3$ where $L(t) = \langle \lambda_1, \lambda_2, \lambda_3 \rangle$

(e) $\bigcup_{i=0}^{w} U_i$ must form a connected set in $T(G, \tau(G))$

Proposition Properties (a), (b), (c), (d), (e) can be expressed in MSO over $\tau = \{ S_1, S_2, <, \{Q_a\}_{a \in \Sigma} \}$

$$\varphi_a(X_0, X_1, \ldots X_w) =$$

$$\bigwedge_{i \neq j} \bigwedge_{\substack{a = (\lambda_1, \lambda_2, \lambda_3) \\ (i,j) \in \lambda_2}} \forall x \; X_i(x) \wedge Q_a(x) \rightarrow X_j(x)$$

Invariant

$$G \models \psi(x_1, x_2 \ldots x_k, Y_1, \ldots Y_\ell)\,[x_i \mapsto u_i,\ Y_j \mapsto S_j]$$

iff

$$T(G, \tau(G)) \models t(\psi)(\bar{X}_1, \bar{X}_2 \ldots \bar{X}_k, \bar{Y}_1, \ldots \bar{Y}_k)$$
$$[\bar{X}_i \mapsto \mathrm{Tree}(u_i),\ \bar{Y}_j \mapsto \mathrm{Tree}(S_j)]$$

Build $t(\psi)$ by structural induction.

$\psi = x = y$: $\bigwedge_{i=0}^{\omega}$ ~~" $X_i = Y_i$ "~~ $\forall z\ X_i(z) \leftrightarrow Y_i(z)$

$z \hookrightarrow$ nodes of T

$x \downarrow \bar{X}$ $\quad y \downarrow \bar{Y}$

$\psi : Y(x)$: $\bigwedge_{i=0}^{\omega}$ " $X_i \subseteq Y_i$ "

$\psi : E\,xy$: $\bigvee_{i,j} \bigvee_{\substack{a=(\lambda_1,\lambda_2,\lambda_3) \\ (i,j)\in\lambda_1}} \exists t\ X_i(t) \wedge Y_j(t) \wedge Q_a(t)$

$\psi : \psi_1 \wedge \psi_2$: $t(\psi_1) \wedge t(\psi_2)$

$\psi : \exists x\ \psi'$: $\exists \bar{X}\ \ t(\psi')(\bar{X}) \wedge$
$\varphi_a(\bar{X}) \wedge \ldots \wedge \varphi_e(\bar{X})$

$\psi : \exists Y\, \psi'$: $\exists \bar{Y}\ \ t(\psi')(\bar{Y}) \wedge$
$\varphi_a(\bar{Y}) \wedge \varphi_b(\bar{Y})$.