Courcelle's Theorem

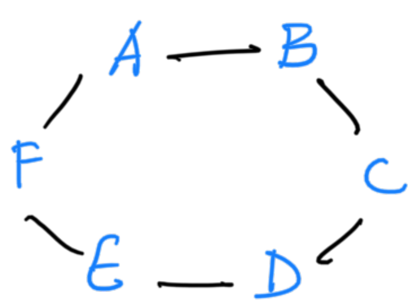
Tree Decomposition of width ω of a graph $G=(V,E)$ is a 2^V-labeled tree $T=(V_T, E_T, L_T)$ such that for every $t \in V_T$, $L_T(t) \subseteq V$ with $|L_T(t)| \leq \omega+1$ satisfying the following properties.

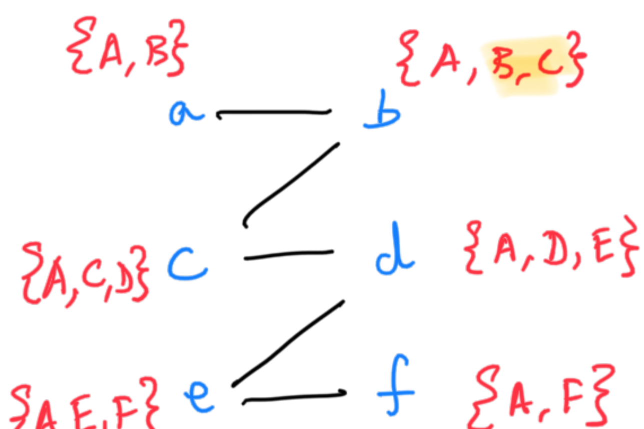
Node Coverage $\forall u \in V \ \exists t \in V_T \ \ u \in L_T(t)$

Edge Coverage $\forall \{u,v\} \in E \ \exists t \in V_T$
$\{u,v\} \subseteq L_T(t)$.

Coherence $\forall x, y \in V_T$ such that $\exists u \in V$
$u \in L_T(x) \cap L_T(y)$, $u \in L_T(z)$ for every vertex z on the path from x to y in the tree T.

Tree width of G $= \min\limits_{T \text{ is a tree decomposition of } G} \text{width}(T)$





Theorem: For any graph $G=(V,E)$, a

Theorem For any graph $G = (V, E)$, a tree decomposition T of G with smallest width can be found in time $O(2^{p(w)} |V|)$ (p - polynomial fn)

- T is binary tree
- # vertices in $T \leq |V|$

Courcelle's Theorem For any MSO sentence φ over signature $\tau = \{E\}$, the problem of determing if $G \models \varphi$ can be solved in time $O(f(|\varphi|, w)|G|)$ where $w = \text{tree width}(G)$.

Idea Reduce the problem of checking if $G \models \varphi$ to the problem of checking if a tree $T_G \models \varphi'$ for some MSO sentence φ'.

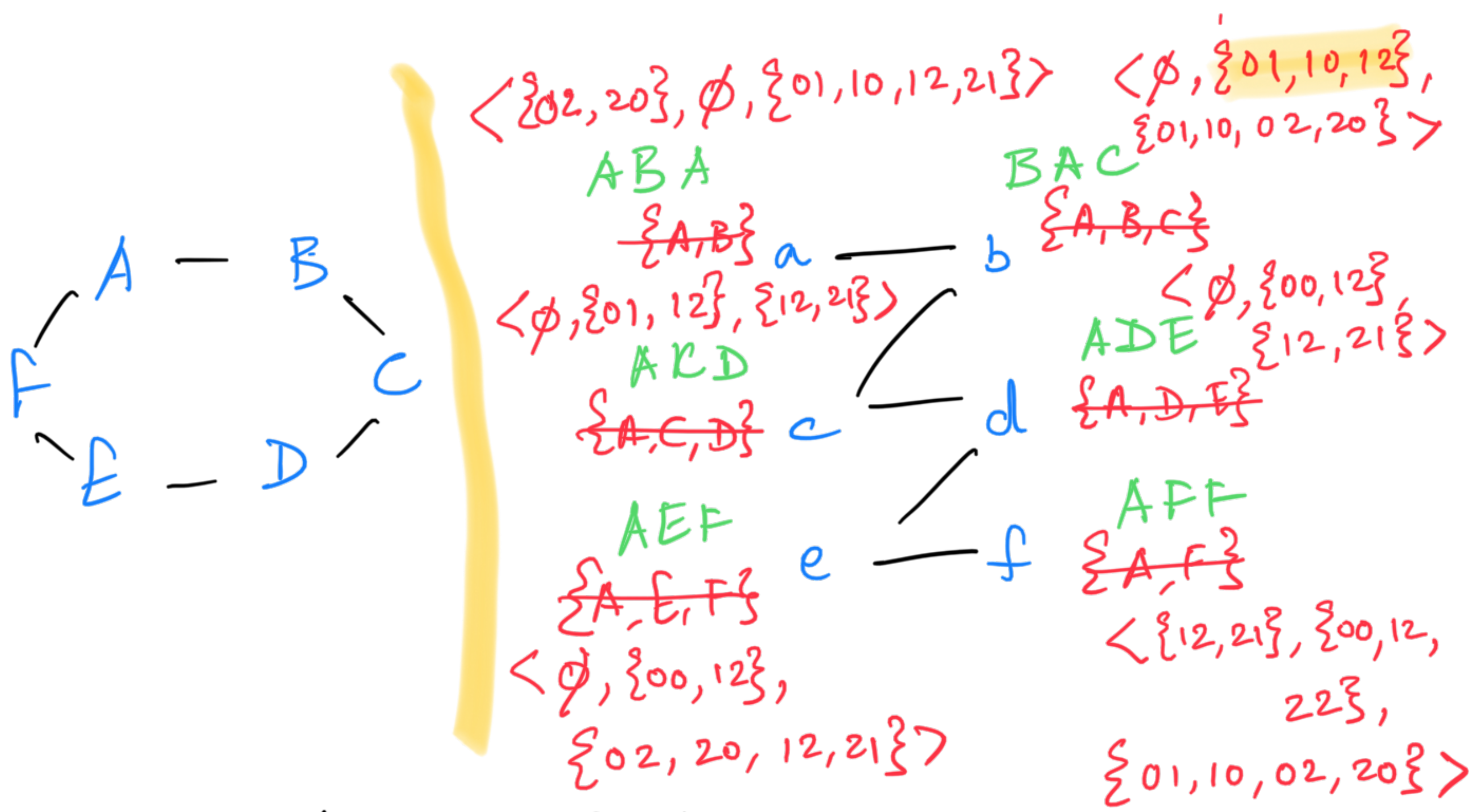
Can be solved by constructing automaton and running on T_G.

What should the tree T_G be?

Take T_G to be the tree decomposition

— Take T_G to be the tree decomp
of G.

— The problem is that the label set of T_G depends on G.



Tree(B) = $\langle \{b\}, \{a\}, \emptyset \rangle$ Tree(A) = $\langle \{a,c,d,e,f\}, \{b\}, \{a\} \rangle$

$\langle \{b\}, \emptyset, \emptyset \rangle$

For a tree decomposition of w, we will about the labels as a string of length $w+1$ over V (vertex of G)

Given a tree decomposition $T = (V_T, E_T, L_T)$

$L_T : V_T \to V^{w+1}$ (graph $G = (V,E)$)

$T(G) = (V_G, E_G, L_G)$

$V_G = V_T$, $E_G = E_T$.

$$L_G : V_G \to \left(\{0,1,2,\dots w\} \times \{0,1,2\dots w\}\right)^3$$

$$L_G(t) = (\lambda_1, \lambda_2, \lambda_3)$$

- $(i,j) \in \lambda_1$ iff $L_T(t)(i) = L_T(t)(j)$
- $(i,j) \in \lambda_2$ iff $L_T(t)(i) = L_T(x)(j)$ where x is parent of t.
- $(i,j) \in \lambda_3$ iff $\{L_T(t)(i), L_T(t)(j)\} \in E$

Proof Idea: Given G of width w and φ. Construct T_G (obtained from tree decomposition of G of width w) and ψ s.t.

$$\underbrace{G \models \varphi}_{\substack{\text{Variables for vertices in } G\text{, Edge of } G \\ \text{Quantifies over vertices and sets of vertices of } G}} \quad \text{iff} \quad \underbrace{T_G \models \psi}_{\substack{\text{Variables refer to vertices } T_G\text{, sets of vertices of } T_G, \\ \text{Edge in } T_G\text{, Labels of vertices in } T_G}}.$$

Encode nodes / sets of nodes of G as vertices / sets of vertices of T_G.

$u \in V \longmapsto w{+}1$-tuple of sets of vertices of T_G.

$\text{Tree}(u) : \langle U_0, U_1, U_2 \dots U_w \rangle$

$$U_i = \{ t \mid L_T(t)[i] = u \}$$

For $X \subseteq V$,

$Tree(X) = \langle U_0 \ldots U_w \rangle$

$$U_i = \{ t \mid L_T(t)[i] \in X \}$$

A tuple $\langle U_0, U_1 \ldots U_w \rangle$ corresponds to $Tree(X)$ of $X \subseteq V$ iff

(a) $\forall\, t \in U_i$ with label $(\lambda_1, \lambda_2, \lambda_3)$
$(i,j) \in \lambda_1 \Rightarrow t \in U_j$

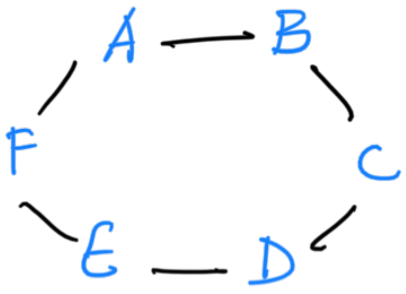
(b) $\forall\, t \in U_i$ with label $(\lambda_1, \lambda_2, \lambda_3)$
$(i,j) \in \lambda_2$ and x is the parent of t.
$\Rightarrow p \in U_j$.

In addition when $|X| = 1$

(c) If $t \in U_i \cap U_j$ with label $(\lambda_1, \lambda_2, \lambda_3)$
then $(i,j) \in \lambda_1$.

(d) If $t \in U_i$ and parent of $t = x \in U_j$
and label of t is $(\lambda_1, \lambda_2, \lambda_3)$ then
$(i,j) \in \lambda_2$.

(e) $\bigcup_{i=0}^{w} U_i$ must form a connected component of T_G.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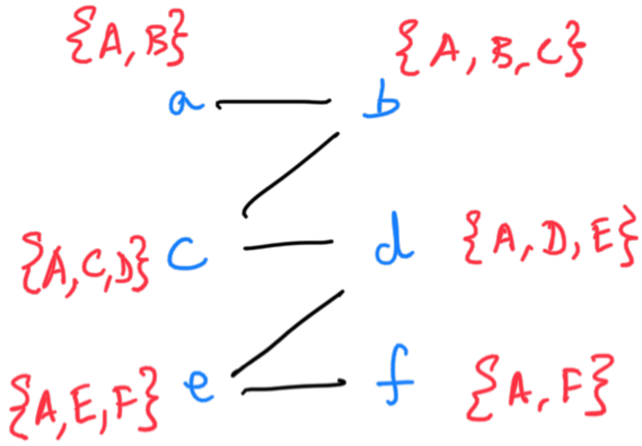
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