Tree Decompositions and Tree Width

Σ-labeled ordered n-ary trees Structures over the signature

$$\tau_T = \left(<, \{S_i\}_{i=0}^{n-1}, \{Q_a\}_{a \in \Sigma} \right)$$

Doner, Thatcher-Wright Theorem The set of Σ-labeled trees definable in MSO is exactly the set of regular tree languages.

Corollary Consider a MSO sentence φ. Given any Σ-labeled tree T, the decision problem of determining if $T \models \varphi$ is decidable in $O(|T|)$.

Proof Construct a tree automaton $\mathcal{A}_\varphi$ corresponding to φ. Expensive

On an input T, run $\mathcal{A}_\varphi$ on T.
linear time

MSO on Graphs.

Signature $\tau_E = \{E\}$

3-colorability A graph $G = (V, E)$ is 3 colorable if $\exists\, c : V \to \{1, 2, 3\}$ s.t.
$\forall u, v: \{u, v\} \in E,\ c(u) \neq c(v)$.

$\forall u,v \quad \{u,v\} \ldots$

Theorem 3-colorability is NP-complete.

Define 3-colorability

$$\varphi_{3col} = \exists X_1 \exists X_2 \exists X_3$$

"every vertex belongs to exactly one set".

$$\wedge \forall x \forall y \; Exy \to \bigwedge_{i=1}^{3} (\neg X_i x \vee \neg X_i y)$$

Independent Set is $I \subseteq V$ in $G = (V, E)$ such that $\forall u, v \in I$, $\{u,v\} \notin E$.

Max Independent Set Given a graph $G = (V, E)$ and $k \in \mathbb{N}$, determine if there is an independent set of size $\geq k$.

– NP-complete.

$$\varphi_{ind} = \exists I \; \exists x_1 \exists x_2 \ldots \exists x_k$$

$$\bigwedge_{i \neq j} \neg (x_i = x_j) \wedge \bigwedge_{i=1}^{n} I x_i$$

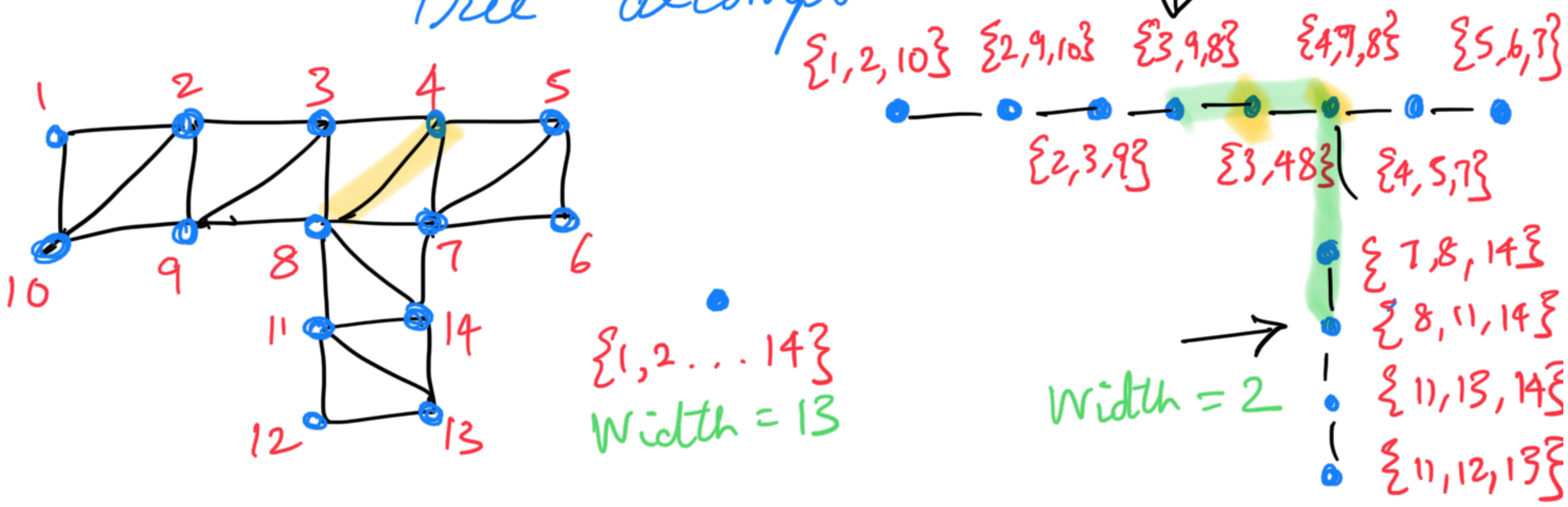
$$\wedge \forall x \forall y. \; Ix \wedge Iy \to \neg Exy$$

Question Can these NP-complete problems be solved efficiently on more general graphs than just trees?

– Can we solve problems definable in ... more general than

MSO on structures ... trees ?

Tree decomposition



For a graph $G = (V, E)$, a tree decomposition is a 2^V-labeled tree $T = (V_T, E_T, L_T)$

Node coverage $\forall u \in V, \ \exists t \in V_T$
$u \in L_T(t)$

Edge coverage $\forall \{u,v\} \in E \ \exists t \in V_T$
$\{u,v\} \subseteq L_T(t)$

Coherence $\forall u \in V$ if $\exists x, y \in V_T$ $x \neq y$ such that $u \in L_T(x) \cap L_T(y)$ then $\forall z$ that appears on the unique path from x to y in T, $u \in L_T(z)$

- Every graph has a trivial tree-decomposition
- A graph may have many tree-decompositions

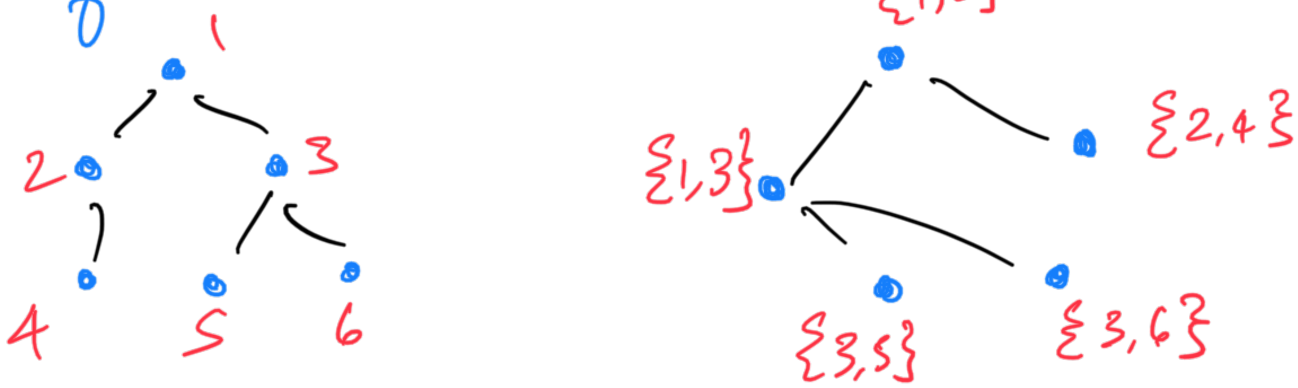
Definition A tree decomposition $T = (V_T, E_T, L_T)$

of graph $G=(V,E)$ has width $w \in \mathbb{N}$
iff $\forall t \in V_T,\ |L_T(t)| \leq w+1$

Tree Width of graph $G=(V,E)$ is w if there is a tree decomposition of G of width $\leq w$.

Proposition Every tree $G=(V,E)$ has a tree decomposition of width 1.

Proof $T=(V_T, E_T, L_T)$



$V_T = E$

$L_T(e=\{u,v\}) = \{u,v\}$

$(e_1, e_2) \in E_T$ if $L_T(e_1) \cap L(e_2) \neq \emptyset$

Proposition If G is a connected graph of width 1 then G is a tree.

Constructing Tree Decompositions of small width.

Problem Given a graph $G=(V,E)$ and ... G has tree width $\leq w$.

$w \in \mathbb{N}$, determine if G has tree width ≤ w.

– NP-complete.

Bodlaender Theorem Given a graph $G = (V, E)$ there is an algorithm A that computes a tree decomposition of smallest width of G in time $T(f(w) \, poly(|V|))$ where w is the tree width of G.

Let $T = (V_T, E_T, L_T)$ be a tree decomposition.

An edge $(t_1, t_2) \in E_T$ is redundant if $L_T(t_1) \subseteq L(t_2)$.

A tree decomposition is non-redundant if it has no redundant edges.

Proposition Every graph G of width w has a non-redundant tree decomposition of width w.

– Redundant edge $\{t_1, t_2\}$ can be contracted – [remove vertices t_1, t_2 and add a vertex $\{t_1, t_2\}$]

Proposition Every graph G of width w has

Proposition Every graph of [illegible]
a tree decomposition of width w that is binary.