Descriptive Complexity II

Logic capturing complexity class.

Logic $\mathcal{L}$ is said to capture complexity class $\mathcal{C}$ if

(a) $\forall \varphi \in \mathcal{L}$ over signature τ, the problem of determining if $\mathcal{A} \models \varphi$ for a given τ-structure $\mathcal{A}$ is in $\mathcal{C}$

(b) For any signature τ and any collection of τ-structures $\mathcal{K}$ that is closed under isomorphism,

$\forall \mathcal{A}, \mathcal{B} \quad \mathcal{A} \cong \mathcal{B} \Rightarrow \mathcal{A} \in \mathcal{K} \leftrightarrow \mathcal{B} \in \mathcal{K}$

$\text{encode}'(\mathcal{K}) = \{ \text{encode}(\mathcal{A}, <) \mid \mathcal{A} \in \mathcal{K} \}$

and $\mathcal{K} \in \mathcal{C}$, there is a τ-sentence $\varphi \in \mathcal{L}$ such that

$$\mathcal{K} = \{ \mathcal{A} \mid \mathcal{A} \models \varphi \}.$$

Existential Second Order Logic

Second order Logic formulas of the form

$$\exists X_1 \exists X_2 \ldots \exists X_m \, \psi$$

where ψ is a first order logic formula.

Fagin's Theorem Existential second order logic captures NP.

Proof Complexity of the problem of checking finite $\mathcal{A} \models \varphi$, $\varphi \in \text{ExSO}$.

$\varphi = \exists X_1 \exists X_2 \ldots \exists X_m \psi$

WLOG $|u(\mathcal{A})| = n$.

- Guess relational interpretations for $X_1, X_2 \ldots X_m$. → poly time
- Check if ψ holds $(\mathcal{A}, X_1 \ldots X_m)$
  FO.

If max arity $X_1 \ldots X_m$ is k then guess $m\,(n^k)$ constants.

Algo $\in$ NP.

Second direction: To show every problem in NP can be defined in ExSO.

$B \in NP$

B is recognized M.

Corollary Universal SO captures coNP

$\forall X_1 \forall X_2 \ldots \forall X_m \psi$

FO.

$\text{coNP} = \{ B \mid \overline{B} \in NP \}$.

What is the relative expressive power

What is the [illegible]
of existential SO and universal SO?
— Equivalent $NP \stackrel{?}{=} coNP$.

How does the expressive power full SO compare to NP?
— Unresolved.
— Corollary Fagin's Theorem SO captures Polynomial Time Hierarchy.

Spectrum $S \subseteq \mathbb{N}$ is a spectrum if there is a signature τ and τ-sentence φ such that

$$S = \{ |\mathcal{A}| \mid \mathcal{A} \models \varphi \}.$$

Scholtz's Question Is the complement of a spectrum also a spectrum?

$$\text{spectrum}(\neg\varphi) \neq \overline{\text{spectrum}(\varphi)}$$

Correspondence between spectra and models of ExSO.

Consider FO φ over τ.

WLOG τ has constants and relations

$$\tau = (c_1 \ldots c_m, R_1 \ldots R_k)$$

spectrum (φ) =
Models $(\exists R_1 \ldots \exists R_\ell \exists c_1 \ldots \exists c_m \varphi)$
ExSO sentence over $\{\}$.

Spectrum = Models of ExSO over $\{\}$.

If S is a spectrum $\Rightarrow$
ExSO $\underline{\underline{\varphi}}$ $\Rightarrow$ $S \in NEXP$

$S \in NEXP \Rightarrow$ Spectrum of $\varphi = S$.

Jones-Selman Theorem S is spectrum iff $S \in NEXP$.

Scholtz's Question is equivalent
$NEXP = co\,NEXP$

Logical Characterization of Polynomial Time.

First order logic is too weak
- Evenness is not expressible.

Ordered Signature $< \in \tau$.

Ordered Structure a structure $\mathcal{A}$ over an ordered signature such that
$<^{\mathcal{A}}$ is a linear order

$\angle$ is a linear order.

L captures $\mathcal{C}$ over ordered structures

- $\forall \varphi$ over an ordered signature the problem $\mathcal{A} \models \varphi \in \mathcal{C}$.
- $\forall$ any ordered signature τ and collection of τ-structures $\mathcal{K}$ closed under isomorphism and $\mathcal{K} \in \mathcal{C}$, $\mathcal{K}$ is definable in L.

Connectivity over ordered graphs in not FO-expressible.

Definition Let $\psi(R, \bar{x})$ where R is s-ary relation and $\bar{x} = \{x_1 \ldots x_s\}$ and ψ is $\tau \cup \{R\}$-formula.

For any τ-structure $\mathcal{A}$, ψ defines

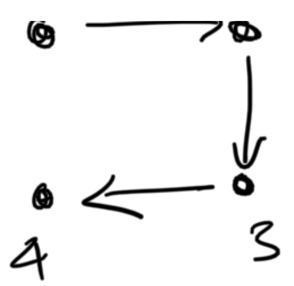
$$F_\psi : 2^{(u(\mathcal{A})^s)} \to 2^{u(\mathcal{A})^s}$$

$$F_\psi(\underset{\subseteq u(\mathcal{A})^s}{T}) = \{\bar{a} \mid \mathcal{A} \models \psi[R \mapsto T, \bar{x} \mapsto \bar{a}]\}$$

Example $\tau = \{E\}$

$$\psi(R, x, y) = E(x,y) \vee (\exists z\; R(x,z) \wedge E(z,y))$$

$F_\psi(\emptyset) = \{(1,2),(2,3),(3,4)\}$
$= E$



$F_\psi(E) = \{(1,2),(2,3),(3,4),$
$(1,3),(2,4)\}$

Fixed Point Suppose $F_\psi : 2^{u(\mathcal{A})^s} \to 2^{u(\mathcal{A})^s}$

$T \subseteq u(\mathcal{A})^s$ is a fixpoint of F_ψ if

$$F_\psi(T) = T.$$

$$F_\psi(\{1,2,3,4\} \times \{1,2,3,4\})$$
$$= \{1,2,3,4\} \times \{1,2,3,4\}$$

$$F_\psi(\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\})$$
$$= \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$$

Least fixpoint $F_\psi : 2^{u(\mathcal{A})^s} \to 2^{u(\mathcal{A})^s}$

T is LFP if T is fixpoint and for any other fixpoint R, we have

$$T \subseteq R.$$

Tarski–Knaster Theorem If $\psi(R, \bar{x})$ $\tau \cup \{R\}$ formula such that all atomic subformulas involving R are under an even number of negations, then F_ψ has a least fixed point.

First Order Logic with LFP.

- If $\psi(R, \bar{x})$ is a formula such that R appears under an even number of negations. then

$$LFP\ R\bar{x}\ \psi(\bar{y}) \text{ is a formula.}$$

Meaning $\mathcal{A} \models LFP\ R\bar{x}\ \psi\ (\bar{y})\ [\bar{y} \mapsto \bar{a}]$

if $\bar{a} \in$ least fixpoint of F_ψ

Immerman Vardi Theorem Over ordered structures FO + LFP captures P.