Logic capturing complexity class.

Logic $L$ is said to capture complexity class $C$ if

(a) $\forall \varphi \in L$ over signature $\Sigma$, the problem of determining if $A \models \varphi$ for a given $\Sigma$-structure $A$ is in $C$.

(b) For any signature $\Sigma$ and any collection of $\Sigma$-structures $K$ that is closed under isomorphism, $\forall A, B, A \cong B \Rightarrow A \in K \iff B \in K$, encode $(K) = \{\text{encode}(A, \preceq) : (A, \preceq) \in K\}$, there is a $\Sigma$-sentence $\varphi \in L$ such that

$$K = \{A : A \models \varphi\}.$$

**Existential Second Order Logic**

Second order logic formulas of the form

$$\exists x_1 \exists x_2 \ldots \exists x_m \varphi$$

where $\varphi$ is a first order logic formula.

**Fagin's Theorem** Existental second order logic captures NP.
Proof: Complexity of the problem of checking finite $A \not\models \varphi$, $\varphi \in \text{ExSo}$.

$\varphi = \exists X_1 \exists X_2 \ldots \exists X_m \psi$

WLOG $(u(A)) = n$.

- Guess relational interpretations for $X_1, X_2 \ldots X_m$. $\rightarrow$ poly time

- Check if $\psi$ holds $(A, X_1 \ldots X_m)$ for.

If max arity $X_1 \ldots X_m$ is $k$ then guess $m(n^k)$ is constant.

Algorithm $\in \text{NP}$.

Second direction: To show every problem in $\text{NP}$ can be defined in $\text{ExSo}$.

$B \in \text{NP}$

$B$ is recognized $M$.

Corollary: Universal $\text{SO}$ captures $\text{coNP}$

$\forall X_1 \forall X_2 \ldots \forall X_m \psi$

$\not\models \text{Fo.}$

$\text{coNP} = \sum B \mid \overline{B} \in \text{NP}$.

$\therefore$ $\therefore$ the relative expressive power
What is the relation of existential $\exists_0$ and universal $\forall_0$?

- Equivalent $\text{NP} \equiv \text{coNP}$.

How does the expressive power fall $\exists_0$ compare to $\text{NP}$?

- Unresolved.

- Corollary: Fagin’s theorem $\exists_0$ captures Polynomial Time Hierarchy.

**Spectrum** $S \subseteq \mathbb{N}$ is a spectrum if there is a signature $\Sigma$ and $\Sigma$-sentence $\psi$ such that

$$S = \{ \bar{a} | \forall \bar{b} \in A \not\models \psi \}.$$

**Schötz’s Question** Is the complement of a spectrum also a spectrum?

**Spectrum** $C(\neg \phi) \neq \text{specturm}(\phi)$

Correspondence between spectra and models of $\text{E}x\text{SO}$. Consider $\text{FO}$ $\phi$ over $\Sigma$.

WLOG $\Sigma$ has constants and relations

$$\Sigma = \{ c_1, \ldots, c_m, R_1, \ldots, R_k \}.$$
Spectrum ($\mathcal{S}$) =

Models ($\forall R_1 \ldots \forall R_e \exists c_1 \ldots \exists c_m \varphi$)

$\exists \sigma$ so sentence over $\mathcal{S}$.

Spectrum = Models of $\exists \sigma$ so over $\mathcal{S}$.

If $S$ is a spectrum $\Rightarrow$

$\forall \sigma \exists \sigma \sigma \varphi \Rightarrow S \in \text{NEXP}$

$S \in \text{NEXP} \Rightarrow$ Spectrum of $\varphi$. = $S$.

Jones-Selman Theorem $S$ is spectrum if $S \in \text{NEXP}$.

Scholtz's Question is equivalent

NEXP = co NEXP

Logical Characterization of Polynomial Time:

First order logic is too weak

- Evenness is not expressible.

Ordered Signature $\langle e \rangle$

Ordered Structure a structure $\mathcal{A}$ over an ordered signature such that
$L$ is a unary logic.

$L$ captures $C$ over ordered structures

- If $\varphi$ over an ordered signature
  the problem $A \models \varphi \in C$.
- If any ordered signature $\Sigma$ and
  collection of $\Sigma$-structures $\mathcal{K}$ closed
  under isomorphism and $\mathcal{K} \subseteq C$,
  $\Phi$ is definable in $L$.

Connectivity over ordered graphs in logic
$\mathcal{L}_o$-expressible.

Definition: Let $\psi(R, \overline{x})$ where
$R$ is $s$-ary relation and $\overline{x} = \overline{x}_1, \ldots, \overline{x}_s$ and
$\psi$ is $\Sigma \cup \exists R \Sigma$-formula.

For any $\Sigma$-structure $A$, $\psi$ defines

$F_{\psi} : 2^{\Sigma(A)^s} \rightarrow 2^{\Sigma(A)^s}$

$F_{\psi}(\overline{T}) = \{ \overline{a} \mid A \models \psi[R \mapsto T, \overline{x} \mapsto \overline{a}] \}$

Example: $C = \{ E \}$

$\psi(R, x, y) = E(x, y) \lor (\exists z \ R(x, z) \land E(z, y))$

$L \subseteq C$ over ordered structures

- If $\varphi$ over an ordered signature
  the problem $A \models \varphi \in C$.
- If any ordered signature $\Sigma$ and
  collection of $\Sigma$-structures $\mathcal{K}$ closed
  under isomorphism and $\mathcal{K} \subseteq C$,
  $\Phi$ is definable in $L$.
\[ F_\psi(\emptyset) = 2 \{ (1,2), (2,3), (3,4) \} \]
\[ F_\psi(E) = \{ (1,2), (2,3), (3,4), (1,3), (2,4) \} \]

**Fixed Point** Suppose \( F_\psi : 2^u \xrightarrow{\text{u}} 2^u \) is a fixpoint of \( F_\psi \) if
\[ F_\psi(T) = T. \]

\[ F_\psi(\emptyset, \{ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \}) \]
\[ = \emptyset, \{ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \} \]

**Least fixpoint** \( F_\psi : 2^u \xrightarrow{\text{u}} 2^u \)

\( T \) is LFP if \( T \) is fixpoint and for any other fixpoint \( R \), we have
\[ T \subseteq R. \]

**Tarski–Knaster Theorem** If \( \psi(\emptyset, x) \)
\[ \exists x \, R^x \] formulas such that all atomic subformulas involving \( R \) are under an even number of negations,
then \( F_\psi \) has a least fixed point.
First Order Logic with LFP.

- If $\psi(R,\bar{x})$ is a formula such that $R$ appears under an even number of negations, then
  $$\text{LFP } R\bar{x} \psi(Cy)$$
is a formula.

  Meaning $A \models \text{LFP } R\bar{x} \psi(Cy) [Cy \mapsto \bar{a}]$

  if $\bar{a}$ is least fixpoint of $F_{\psi}$

Immerman-Vardi Theorem: Over ordered structures $F_0 + \text{LFP}$ captures $\mathcal{P}$. 