Descriptive Complexity

Ehrenfeucht's Theorem The following are equivalent statements

(a) For every formula $\varphi(x_1 \ldots x_s)$ s.t $qr(\varphi) \leq m$

$\mathcal{A} \models \varphi[x_i \mapsto a_i]$ iff $\mathcal{B} \models \varphi[x_i \mapsto b_i]$

(b) $\mathcal{B} \models \varphi_m^{\mathcal{A},\bar{a}} [x_i \mapsto b_i]$ → Scott Hintikka

(c) D wins the m-round game $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$

Definition Let τ be some signature and K is a collection of (finite) τ-structures. K is <u>definable</u> if there is τ-sentence φ such that

$$K = \{\mathcal{A} \mid \mathcal{A} \models \varphi\}.$$

Theorem Let K be a set of τ-structures. K is definable iff $\exists m \in \mathbb{N}$ s.t. $\forall \mathcal{A}, \mathcal{B}$. if $\mathcal{A} \equiv_m \mathcal{B}$ then $\mathcal{A} \in K$ iff $\mathcal{B} \in K$.

$\downarrow$

$\forall \varphi.\ qr(\varphi) \leq m.\ \mathcal{A} \models \varphi \Leftrightarrow \mathcal{B} \models \varphi.$

Proof ($\Rightarrow$) Let φ be s.t.

$$K = \{\mathcal{A} \mid \mathcal{A} \models \varphi\}.$$

Let $m = qr(\varphi)$. Let $\mathcal{A} \equiv_m \mathcal{B}$

$$\mathcal{A} \in K \Rightarrow \mathcal{A} \models \varphi \Rightarrow \mathcal{B} \models \varphi \Rightarrow \mathcal{B} \in K.$$

($\Leftarrow$) $\exists m$ s.t. $\forall\ \mathcal{A} \equiv_m \mathcal{B}$. $\mathcal{A} \in K \Leftrightarrow \mathcal{B} \in K$.

$$\varphi = \bigvee_{\mathcal{A}\in K} \varphi_m^{\mathcal{A}}$$

Suppose $\mathcal{B} \models \varphi \Rightarrow \exists \mathcal{A}\in K \ \ \mathcal{B}\models \varphi_m^{\mathcal{A}}$

$\Rightarrow \mathcal{B} \equiv_m \mathcal{A} \Rightarrow \mathcal{B}\in K$

Suppose $\mathcal{B}\in K \Rightarrow \mathcal{B}\models\varphi_m^{\mathcal{B}}$

$\varphi_m^{\mathcal{B}} \models \varphi$

$\Rightarrow \mathcal{B}\models\varphi.$

Ordered Signature $\tau_0 = \{<\}$.

Ordered Structure τ_0 - structure.

Proposition Let

$$Even = \{\mathcal{A} \mid |u(\mathcal{A})| \text{ is even}\}.$$

Even is not definable.

Proof

Gurevich's Theorem For any ordered $\mathcal{A}, \mathcal{B}$ $|u(\mathcal{A})| \geq 2^m$ and $|u(\mathcal{B})| \geq 2^m$. Then D wins $G_m(\mathcal{A}, \mathcal{B})$.

For any m, $|u(\mathcal{A})| = 2^m$ and $|u(\mathcal{B})| = 2^m + 1$

$\mathcal{A} \equiv_m \mathcal{B}$

But $\mathcal{A} \in Even$ and $\mathcal{B} \notin Even$.

Ordered Graphs $\tau_{OG} = \{<, E\}$. Ordered graphs are τ_{OG} - structures.

Connected Graphs. For every pair of vertices u, v there is a path from u to v.

Theorem Over ordered graphs connectivity is not definable. i.e.

$\text{Connected} = \{\mathcal{A} \mid \mathcal{A} \text{ is } \tau_{OG}\text{-structures such that } \mathcal{A} \text{ is connected}\}$.

Connected is not definable.

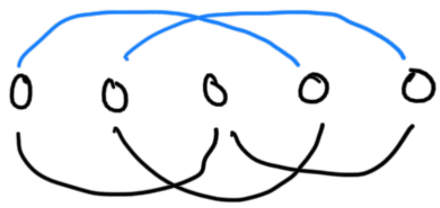
Proof Suppose (for contradiction) φ_{con} defines Connected

$\psi_E(x,y) = \begin{cases} x \,\&\, y \text{ are 2 apart} \\ \text{or } x \text{ is first } \& \ y \text{ is second-last or } x \text{ is second } \& \ y \text{ is last} \end{cases}$

Not connected

$\text{first}(x) = \forall y \ \neg(y < x)$

$\text{last}(x) = \forall y \ \neg(x < y)$



Connected

$\text{second}(x) = \forall y. \ (y < x) \to \text{first}(y)$

Observe $\mathcal{A}$ is even iff

$$\mathcal{A} \models \neg\varphi_{con}(E \mapsto \psi_E)$$

Contradiction.

Descriptive Complexity

Logic to capture a complexity class.

L captures $\mathcal{C}$. iff

(as $\forall \varphi \in L$ signature τ τ $\{\mathcal{A} \mid \mathcal{A} \models \varphi\} \in \mathcal{C}$.

(b) $\forall$ signature τ collection of τ structures $K \in \mathcal{C}$

$\exists$ τ-sentence $\varphi \in L$. s.t $K = \{\mathcal{A} \mid \mathcal{A} \models \varphi\}$.

Encode Structures as strings

Encoding of Graph (V, E)

- assuming V are $\{1, 2 .. n\}$. Ordered
- $E = \{1, .. n\} \times \{1 .. n\}$

$$E(i,j) = \begin{cases} 1 & \text{if edge from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$

Binary string $\underbrace{\sqcup \sqcup \sqcup \sqcup}_{\text{Edges out of } 1} \sqcup \ . \ . \ . \ .$

1 iff $(i,j) \in E$

Typically encoding of $\mathcal{A}$ is an encoding of $(\mathcal{A}, <)$ $\left\{ \begin{array}{l} \text{encode}(\mathcal{A}, <) \\ \neq (\mathcal{A}, <') \end{array} \right.$

- $(\mathcal{A}, <) \cong (\mathcal{B}, <)$ then $\text{encode}(\mathcal{A}, <) = \text{encode}(\mathcal{B}, <)$
- encoding are polynomial long.
- Checking if a relation holds on some elements given an encoding.
-

$K \in \mathcal{C}$ mean

$\text{encode}(K) = \{\text{encode}(\mathcal{A}, <) \mid \mathcal{A} \in K\}$
$\in \mathcal{C}$.

Second Order logic over τ $\left\{ \begin{array}{l} \mathcal{V}_1 - \text{FO variable} \\ \mathcal{V}_2 - \text{Relation variable} \end{array} \right.$

Terms — $x \in \mathcal{V}_1$ $X^k \in \mathcal{V}_2$.

— $c \in \mathcal{C}$

— f is k-ary function, and $t_1 \ldots t_k$ are terms then $f(t_1 \ldots t_k)$ is term

Formulas — t_1, t_2 terms, $t_1 = t_2$

— $t_1 \ldots t_k$ terms $\underset{}{R}(t_1 \ldots t_k)$, $R \in \mathcal{C}$

— $t_1, \ldots t_k$ terms, $X^k \in \mathcal{V}_2$, $X^k(t_1 \ldots t_k)$

— φ is formula then $\neg\varphi$ is formula

— φ, ψ are formulas then $\varphi \vee \psi$

— φ is formula, $x \in \mathcal{V}_1$ then

$\exists x \varphi$

— φ is formula $X^k \in \mathcal{V}_2$ then

$\exists X^k \varphi$.

"there is a k-ary relation X s.t φ holds"

Semantics

Assignments $\alpha = (\alpha_1, \alpha_2)$ for $\mathcal{A}$.

$\alpha_1 : \mathcal{V}_1 \to u(\mathcal{A})$

α_2 meaning to relational variables

For $X^k \in \mathcal{V}_2$

$\alpha_2(X^k) \subseteq u(\mathcal{A})^k$.

$\mathcal{A} \models X^k t \ldots t_k \, [\alpha]$ iff

$(\alpha_1(t_1) \ldots \alpha_1(t_k))^{\mathcal{U}} \in \alpha_2(X^k)$

$\mathcal{A} \models \exists X^k \varphi[\alpha]$ iff $\exists S \subseteq u(\mathcal{A})^k$ s.t

$$\mathcal{A} \models \varphi[\alpha[X^k \mapsto S]].$$

Existential Second Order Logic

Formulas of the form.

$$\exists X_1 \exists X_2 \ldots \exists X_m \psi$$

where ψ is first order.

Fagin's Theorem Existential Second order Logic capture NP.

— $\forall$ τ-sentence φ is Ex SO

$$\{\mathcal{A} \mid \mathcal{A} \models \varphi\} \text{ is in NP}$$

— $\forall$ collection of τ-structures $\mathcal{K} \in NP$

there is an ex SO φ s.t

$$\mathcal{K} = \{\mathcal{A} \mid \mathcal{A} \models \varphi\}.$$