EF Games II

Without loss of generality, assume signature τ only has constants and relation symbols.

Partial Isomorphism A partial isomorphism between τ-structures $\mathcal{A}$ and $\mathcal{B}$ is a partial function $p : u(\mathcal{A}) \hookrightarrow u(\mathcal{B})$ s.t.

(a) $\forall c \in \tau$, $c^{\mathcal{A}} \in dom(p)$ and $p(c^{\mathcal{A}}) = c^{\mathcal{B}}$

(b) p is injective on its domain i.e if $p(x) = p(y)$ then $x = y$.

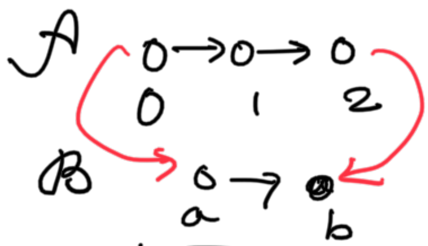
(c) $\forall R \in \tau$ and $a_1 \ldots a_n \in dom(p)$
$(a_1 \ldots a_n) \in R^{\mathcal{A}}$ iff $(p(a_1) \ldots p(a_n)) \in R^{\mathcal{B}}$.

Proposition If p is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$, $\varphi(x_1 \ldots x_n)$ is a quantifier free formula, and α is an assignment such that $\alpha(x_i) \in dom(p)$ $\forall i$, then

$$\mathcal{A} \models \varphi[\alpha] \quad \text{iff} \quad \mathcal{B} \models \varphi[x_i \mapsto p(\alpha(x_i))]$$

Proof Partial isomorphism preserves truth of atomic formulas

Counterexample $\varphi(x, y) = \exists z \; x < z \wedge z < y$



$\mathcal{A} \models \varphi[x \mapsto 0, y \mapsto 2]$ $\quad \mathcal{B} \not\models \varphi[x \mapsto a, y \mapsto b]$

EF Game for m-rounds Played between S (spoiler) and D (duplicator). In each round

- S picks a structure (either $\mathcal{A}$ or $\mathcal{B}$) and an element from the structure
- D responds by picking an element from the other structure.

Let (a_i, b_i) be the elements picked from $\mathcal{A}$ and $\mathcal{B}$, respectively in round i.

D wins the play $(a_1, b_1) \dots (a_m, b_m)$ if $[a_i \mapsto b_i, c^{\mathcal{A}} \mapsto c^{\mathcal{B}}]$ is a partial isomorphism

S wins the play $(a_1, b_1) \dots (a_m, b_m)$ if $[a_i \mapsto b_i, c^{\mathcal{A}} \mapsto c^{\mathcal{B}}]$ is not a partial isomorphism

— D wins game $G_m(\mathcal{A}, \mathcal{B})$ if D has a strategy such that D wins no matter how S plays.

— S wins game $G_m(\mathcal{A}, \mathcal{B})$ if S has a strategy such that S wins no matter how D plays.

Proposition $\mathcal{A} \cong \mathcal{B}$ then D will win $G_m(\mathcal{A}, \mathcal{B})$
($\mathcal{A}$ & $\mathcal{B}$ are isomorphic)

no matter what m.

Proof Let h be the isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

D strategy : When S picks $a \in u(\mathcal{A})$ then
D responds by picking $h(a)$
When S picks $b \in u(\mathcal{B})$ then D
responds by picking $h^{-1}(b)$.

Proposition If D wins $G_m(\mathcal{A}, \mathcal{B})$ then
D wins $G_n(\mathcal{A}, \mathcal{B})$ where $n \leq m$.
If S wins $G_m(\mathcal{A}, \mathcal{B})$ then S wins $G_n(\mathcal{A}, \mathcal{B})$
where $m \leq n$.

Example

$\mathcal{A}$: $a \to b \to c \to d$

$\mathcal{B}$: $1 \to 2 \to 3 \to 4 \to 5$

D wins $G_2(\mathcal{A}, \mathcal{B})$

S wins $G_3(\mathcal{A}, \mathcal{B})$.
- Round 1 S picks 3. D is forced to pick b or c
- S focuses game on the "shorter" side.

$\mathcal{E}$: $a \to b \to c \to d \to e \to f \to g \to h$

$\mathcal{F}$: $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9$.

Claim D wins $G_3(\mathcal{E}, \mathcal{F})$

Gurevich's Theorem Let $\tau_0 = \{<\}$ and
$\mathcal{A}$ and $\mathcal{B}$ are τ_0-structures such that
$|u(\mathcal{A})| \geq 2^m$ and $|u(\mathcal{B})| \geq 2^m$.

Then D wins the $G_m(\mathcal{A}, \mathcal{B})$.

Proof By induction on m.

Base Case : $m=0$ The $\emptyset$ partial function is a partial isomorphism.

Induction Step : Let $\mathcal{A}$ and $\mathcal{B}$ such that $|u(\mathcal{A})| \geq 2^m$ and $|u(\mathcal{B})| \geq 2^m$. $(m>0)$

Let a_{min} and a_{max} be the minimum & maximum (w.r.t $<^{\mathcal{A}}$) in $\mathcal{A}$.

Let b_{min} and b_{max} be the min & max in $\mathcal{B}$.

WLOG, S picks $a \in u(\mathcal{A})$.

- $dist(a_{min}, a) < 2^{m-1} \Rightarrow dist(a, a_{max}) > 2^{m-1}$

  D will pick b s.t $dist(b_{min}, b) = dist(a_{min}, a)$

  $\Rightarrow dist(b, b_{max}) > 2^{m-1}$.

  In subsequent rounds, if S picks $[a_{min}, a]$ or $[b_{min}, b]$, D will respond with ~~the~~ isomorphic element.

  And if S pick $[a, a_{max}]$ or $[b, b_{max}]$ then D will respond according to inductive strategy.

- $dist(a, a_{max}) < 2^{m-1} \Rightarrow dist(a_{min}, a) > 2^{m-1}$

  Works in same way as previous.

- $dist(a_{min}, a) \geq 2^{m-1}$ and $dist(a, a_{max}) \geq 2^{m-1}$

  D will pick any element b such that $dist(b_{min}, b) \geq 2^{m-1}$ $dist(b, b_{max}) \geq 2^{m-1}$

  In subsequent rounds D plays according to the inductive strategy.

Theorem If D wins $G_m(\mathcal{A}, \mathcal{B})$ then for any sentence φ s.t $qr(\varphi) \leq m$.

$\mathcal{A} \models \varphi$ iff $\mathcal{B} \models \varphi$.

m round EF game with starting positions.

$G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ where $\bar{a} = a_1 \dots a_s$ and $\bar{b} = b_1 \dots b_s$. In each round

- S chooses structure & element
- D responds with element in other structure.

Play $(a_1', b_1'), (a_2', b_2') \dots (a_m', b_m')$

D wins if $[a_i \mapsto b_i, a_j' \mapsto b_j', c^{\mathcal{A}} \mapsto c^{\mathcal{B}}]$ is partial isomorphism.

Claim. D wins $G_0(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ iff for any quantifier free formula $\varphi(x_1 \dots x_s)$

$\mathcal{A} \models \varphi[x_i \mapsto a_i] \iff \mathcal{B} \models \varphi[x_i \mapsto b_i]$

Scott-Hintikka Formulas.

Let $\mathcal{A}$ and $\bar{a} \in u(\mathcal{A})^s$.

$$\varphi_0^{\mathcal{A},\bar{a}} = \bigwedge \psi(x_1 \ldots x_s)$$

ψ is atomic or negation of atomic

$\mathcal{A} \models \psi[x_i \mapsto a_i]$

$$\varphi_{m+1}^{\mathcal{A},\bar{a}} = \bigwedge_{a' \in u(\mathcal{A})} \exists x_{m+1} \varphi_m^{\mathcal{A},\bar{a},a'} \wedge$$

$$\forall x_{m+1} \bigvee_{a' \in u(\mathcal{A})} \varphi_m^{\mathcal{A},\bar{a},a'}$$

Proposition $qr(\varphi_m^{\mathcal{A},\bar{a}}) = m$

Second for any s, m, $\{\varphi_m^{\mathcal{A},\bar{a}} \mid \bar{a} \in u(\mathcal{A})^s\}$ is finite set.

Proof By induction on.

Ehrenfeucht's Theorem The following are equivalent statements.

(a) For any $\varphi(x_1 \ldots x_s)$ $qr(\varphi) \leq m$

$\mathcal{A} \models \varphi[x_i \mapsto a_i]$ iff $\mathcal{B} \models \varphi[x_i \mapsto b_i]$

(b) $\mathcal{B} \models \varphi_m^{\mathcal{A},\bar{a}}[x_i \mapsto b_i]$

(c) D wins $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$

Proof (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).