Ehrenfeucht-Fraisse Games

Classical Model Theory

- Gödel's Completeness Theorem
- Compactness Theorem
- Downward Lowenheim-Skolem Theorem

Isomorphism An isomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a bijective function $h : u(\mathcal{A}) \to u(\mathcal{B})$ such that

- $\forall c \in \tau.\ h(c^{\mathcal{A}}) = c^{\mathcal{B}}$
- $\forall f \in \tau.\ h(f^{\mathcal{A}}(a_1 \dots a_n)) = f^{\mathcal{B}}(h(a_1), \dots h(a_n))$
- $\forall R \in \tau\ (a_1 \dots a_n) \in R^{\mathcal{A}}$ iff $(h(a_1) \dots h(a_n)) \in R^{\mathcal{B}}$.

Denote $\mathcal{A} \cong \mathcal{B}$

Elementary Equivalence Structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if

$$Th(\mathcal{A}) = Th(\mathcal{B})$$

or $\forall \varphi\ \ \mathcal{A} \models \varphi$ iff $\mathcal{B} \models \varphi$

Denote this $\mathcal{A} \equiv \mathcal{B}$.

Corollary There are structures $\mathcal{A}, \mathcal{B}$ s.t. $\mathcal{A} \not\cong \mathcal{B}$ and $\mathcal{A} \equiv \mathcal{B}$.

Finite Model Theory

Trakhtenbrot's Theorem Validity in finite structures is not recursively enumerable.

Proposition There is a set sentences Γ such that every finite subset of Γ has a finite model but Γ itself has no finite model.

Theorem For every structure $\mathcal{A}$, there is a sentence $\varphi_{\mathcal{A}}$ such that for any $\mathcal{B}$, $\mathcal{B} \models \varphi_{\mathcal{A}}$ iff $\mathcal{A} \cong \mathcal{B}$.

Proof Let $\mathcal{A}$ such $u(\mathcal{A}) = \{a_1 \dots a_n\}$

$$\varphi_{\mathcal{A}} = \exists x_1 \exists x_2 \dots \exists x_n$$

$$\bigwedge_{i \neq j} \neg (x_i = x_j) \wedge \forall x_{n+1} \bigvee_i (x_{n+1} = x_i)$$

$$\bigwedge_{c \in \tau,\ c^{\mathcal{A}} = a_i} c = x_i$$

$$\bigwedge_{\substack{f, g \in \tau \\ f^{\mathcal{A}}(a_{1_1} \dots a_{1_k}) = g^{\mathcal{A}}(a_{2_1} \dots a_{2_l})}} f(x_{1_1}, x_{1_2} \dots x_{1_k}) = g(x_{2_1} \dots x_{2_l})$$

$$\bigwedge_{\substack{R \in \tau \\ (a_{11} \dots a_{1k}) \in R^{\mathcal{A}}}} R(x_{11} \dots x_{1k}) \wedge \bigwedge_{\substack{R \in \tau \\ (a_{11} \dots a_{1k}) \notin R^{\mathcal{A}}}} \neg R(x_{11} \dots x_{1k})$$

Corollary For any collection $\mathcal{K}$ of finite structures, there is a set of sentences Γ such that for any structure $\mathcal{A}$

$$\mathcal{A} \models \Gamma \text{ iff } \mathcal{A} \in \mathcal{K}.$$

Proof $\eta_{\geq k}$ — structures that have $\geq k$ elements.

$$\eta_{\leq k} = \neg \eta_{\geq k+1} \quad \text{(structures that have} \leq k \text{ elements)}$$

$$\eta_{=k} = \eta_{\geq k} \wedge \eta_{\leq k} \quad \text{(structures with } k \text{ elements)}$$

$$\Gamma = \{ \varphi_\ell \mid \ell \in \mathbb{N} \}$$

$$\varphi_\ell = \eta_{=\ell} \rightarrow \bigvee_{\substack{\mathcal{A} \in \mathcal{K} \\ |U(\mathcal{A})| = \ell}} \varphi_{\mathcal{A}}$$

Question What collections of finite structures can be defined by a single sentence?

Example Classical first order logic.

$\tau_G = \{E, s, t\}$ signature of graphs with vertices s and t.

Undirected graphs.

A graph is **connected** if for every pair of vertices u and v there is a path of finite length from u to v.

Claim First order logic cannot express the fact that s is connected to t.

Proof Assume for contradiction that there is a sentences φ_c that says that the graph is such that s is connected to t

That is $\forall \mathcal{A}.\ \mathcal{A} \models \varphi_c$ iff $\mathcal{A}$ is connected.

ψ_n — There is no path of length n between s and t.

$$= \forall x_1 \forall x_2 \ldots \forall x_{n+1}$$

$$(s = x_1) \wedge (t = x_{n+1}) \bigwedge_{i \neq j} \neg (x_i = x_j) \rightarrow \bigvee_i \neg E(x_i, x_{i+1})$$

$$\Gamma = \{\varphi_c\} \cup \{\psi_n \mid n \in \mathbb{N}\}$$

Γ is finitely satisfiable.

$\Rightarrow$ Γ is satisfiable.

But in the model where Γ is true, s and t are not connected.

Contradiction. So φ_c does not express connectedness.

Expressiveness of FO

Ehrenfeucht - Fraisse Games. Fix structures $\mathcal{A}$ and $\mathcal{B}$ over signature τ.

Game is played between 2 players called Spoiler (S) and Duplicator (D).

m-round E-F games.

In each round,

S picks one of the two structures $\mathcal{A}, \mathcal{B}$. and picks one element from the universe of the picked structure.

D picks an element from the other structure

At the end,

$(a_1, b_1), \ldots (a_m, b_m)$.

D wins this instance of the game if

$$\left[a_1 \mapsto b_1, a_2 \mapsto b_2 \ldots a_m \mapsto b_m, \{c^{\mathcal{A}} \mapsto c^{\mathcal{B}}\}_{c \in \tau}\right]$$

is a partial isomorphism.

Definition A partial function $p : u(\mathcal{A}) \hookrightarrow u(\mathcal{B})$ is a partial isomorphism

- p is injective.
- $\forall c, \quad p(c^{\mathcal{A}}) = c^{\mathcal{B}}$. $\left[\begin{matrix} c^{\mathcal{A}} \in dom(p) \\ \forall c \in \tau \end{matrix}\right]$
- $\forall R \in \tau. \quad \forall a_1 \ldots a_n \in dom(p)$

$(a_1 \ldots a_n) \in R^{\mathcal{A}}$ iff $(p(a_1) \ldots p(a_n)) \in R^{\mathcal{B}}$.

Assuming τ has constants and relation symbols.

Example $\tau_0 = (<)$ — $<$ is ordering relation

$\mathcal{A}$ $\quad a \to b \to c \to d \qquad (\to \text{ is } <)$

$\mathcal{B}$ $\quad 1 \to 2 \to 3 \to 4 \to 5$

$(b, 2), (c, 3), (a, 1)$

$p: a \mapsto 1, b \mapsto 2, c \mapsto 3$ is partial iso.

$(d, 5), (a, 1), (b, 3)$

$(b, 3)\ (a, 1), (\quad 2)$

S wins $G_3(\mathcal{A}, \mathcal{B})$

Definition Duplicator wins the m-round E-F game on $\mathcal{A}, \mathcal{B}$ $[G_m(\mathcal{A}, \mathcal{B})]$ if the Duplicator wins no matter how the Spoiler plays.

S wins $G_m(\mathcal{A}, \mathcal{B})$ if S can play in such way that they win no matter how D plays.

Definition Quantifier rank of a formula

is defined as follows:

$qr(\varphi) = 0$ if φ is atomic formula

$qr(\neg\varphi) = qr(\varphi)$

$qr(\varphi \wedge/\vee \psi) = \max(qr(\varphi), qr(\psi))$

$qr(\exists x\, \varphi) = 1 + qr(\varphi)$.

$$\eta_{\geq k} = \exists x_1 \ldots \exists x_k \bigwedge_{i \neq j} \neg (x_i = x_j)$$

$qr(\eta_{\geq k}) = k$.

$qr(\eta_{\leq k}) = qr(\neg \eta_{\geq k+1}) = k+1$.

$qr(\eta_{=k}) = qr(\eta_{\geq k} \wedge \eta_{\leq k}) = k+1$.

Definition $\mathcal{A} \equiv_m \mathcal{B}$ iff $\forall \varphi$ $qr(\varphi) \leq m$
$\mathcal{A} \models \varphi$ iff $\mathcal{B} \models \varphi$.

Theorem D wins $G_m(\mathcal{A}, \mathcal{B})$ iff
$\mathcal{A} \equiv_m \mathcal{B}$.