Finite Model Theory

Classical Model Theory

Study of mathematical objects (graphs, algebraic structures) through lens logic.

Gödel's Completeness Theorem If Γ is recursively enumerable and φ is a sentence then the problem of determining if $\Gamma \models \varphi$ is recursively enumerable.

- The set of valid sentences is recursively enumerable.

Compactness Theorem A set of sentences Γ is unsatisfiable iff $\exists$ finite $\Gamma_0 \subseteq \Gamma$ such that Γ_0 is unsatisfiable.

Proof By Skolemization Γ is a set universally quantified sentences.

$$\Gamma^* = \{ \psi[x_i \mapsto t_i] \mid \forall x_1 .. \forall x_n \psi \in \Gamma \text{ and } t_1 .. t_n \text{ are ground terms} \}$$

Γ is satisfiable iff Γ^* is satisfiable.

Γ^* unsatisfiable $\Rightarrow \exists$ finite $\Delta \subseteq \Gamma^*$ that is unsatisfiable.

$\Gamma_0 = \{ \varphi \in \Gamma \mid$ some ground instantiation φ of $\varphi \in \Delta \}$

$\Rightarrow \Gamma_0$ is unsatisfiable.

Example

There sentences φ such that

$\forall \mathcal{A} \quad \mathcal{A} \models \varphi \Rightarrow \mathcal{A}$ is finite.

$\varphi = \exists x \, \forall y \;\; x = y$

Proposition There is no sentence φ such that (a) every structure satisfying φ is finite and (b) φ has models of arbitrary size.

Proof Assume φ has finite models of arbitrary size

$$\eta_{\geq k} = \exists x_1 \exists x_2 \ldots \exists x_k \bigwedge_{i \neq j} \neg (x_i = x_j)$$

If $\mathcal{A} \models \eta_{\geq k}$ then $u(\mathcal{A})$ has at least k elements.

$$\Gamma = \{\varphi\} \cup \{\eta_{\geq k}\}_{k \in \mathbb{N}}$$

Every finite subset of Γ is satisfiable.

— φ has models of arbitrary size

From compactness, Γ is satisfiable

Any model of Γ must be infinite.

$\Rightarrow \exists$ infinite model for φ.

Downward Lowenheim-Skolem Theorem

If τ is a countable signature and Γ is a set of τ-sentences that satisfiable then there is a countable structure $\mathcal{A}$ such that $\mathcal{A} \models \Gamma$.

Proposition There are non-isomorphic structures $\mathcal{A}$ and $\mathcal{B}$ such that

$$Th(\mathcal{A}) = Th(\mathcal{B})$$

Proof $\quad Th(\mathbb{R}, <) = Th(\mathbb{Q}, <)$

$$Th(\mathbb{R}, 0, 1, +, <) = Th(\mathbb{Q}, 0, 1, +, <)$$

Finite Model Theory Study of first order logic restricted to finite structures

— φ/Γ is satisfiable in finite models

— φ / T is valid if φ / T' holds in all finite models.

Trakhtenbrot's Theorem The problem of checking if a sentence φ is true in all finite structures is coRE-complete.

Proof FinValid $\in$ coRE

— To check if φ is satisfiable in a finite model: Enumerate all finite models $\mathcal{A}$ and check if $\mathcal{A} \models \varphi$.

FinValid is coRE-hard.

Church Turing Theorem Validity is RE hard.

$$\varphi_{nat} \begin{cases} \forall x \, \exists y \;\; S(x,y) \\ \forall x \, \forall y \, \forall z \;\; S(x,z) \wedge S(y,z) \rightarrow x = y \\ \forall x \, \neg S(x,0) \end{cases}$$

$$\underset{0}{\circ} \xrightarrow{S} \circ \xrightarrow{S} \circ \rightarrow \circ \cdots\cdots$$

$$\cdots\cdots \xrightarrow{S} \circ \xrightarrow{S} \circ \xrightarrow{S} \circ \cdots\cdots$$

φ_{accept}

MP $\leq_m$ Validity.

Constructed φ.

$\neg\varphi_{accept}$

$$\varphi_w = \overbrace{\left(\varphi_{nat} \wedge \varphi_{initial} \wedge \varphi_{const}\right)} \to \varphi_{accept}$$

Can't use these ideas to

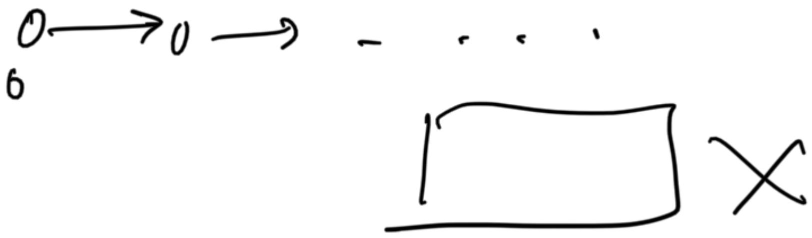
$\overline{MP} \leq$ Validity

Gödel's Incompleteness Theorem

$Th(\mathbb{N}, 0, 1, +, \times, <)$ is not R.E.

$$\overline{HP} \leq Th(\mathbb{N}, 0, 1, +, \times, <)$$

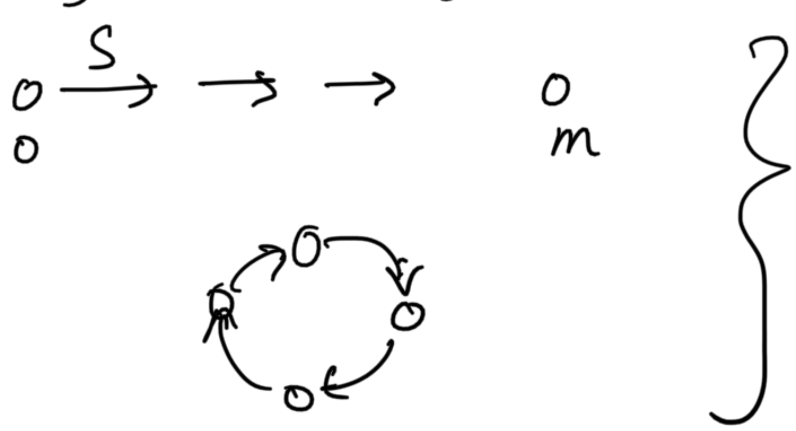
$$\varphi_{\langle M, w\rangle} = \varphi_{initial} \wedge \varphi_{const} \to \neg\varphi_{Halt}.$$



Goal: $\overline{MP} \leq_m$ FinValidity.

Fin Nat
$$\begin{cases} \forall x\, \neg S(x, 0) \\ \forall x \forall y \forall z\ S(x,z) \wedge S(y,z) \to x = y \\ \forall y\ \neg S(m, y) \\ \forall y\ \neg(y = m) \to \exists x\ S(y, x) \end{cases}$$

Finite Models



Given w ...

φ_w is valid in all finite models iff Universal TM U does not accept w.

$$\varphi_w = \varphi_{finNat} \wedge \varphi_{init} \wedge \varphi_{Const} \rightarrow \neg State(m, q_{acc})$$

Compactness Theorem does not hold in finite models.

Proposition There is a set of sentences Γ s.t. every finite subset $\Gamma_0 \subseteq \Gamma$ has a finite model but Γ does not have any finite model.

Proof $\Gamma = \{ \eta_{\geq k} \}_{k \in \mathbb{N}}$

Definition Let τ be some signature. A homomorphism between τ-structures $\mathcal{A}$ and $\mathcal{B}$, $h : u(\mathcal{A}) \to u(\mathcal{B})$ s.t.

- $\forall c \in \tau. \quad h(c^{\mathcal{A}}) = c^{\mathcal{B}}$
- $\forall a_1 .. a_n \in u(\mathcal{A})\ f \in \tau,$
  $h(f^{\mathcal{A}}(a_1 .. a_n)) = f^{\mathcal{B}}(h(a_1) .. h(a_n))$

— $\forall a_1 \ldots a_n \in u(\mathcal{A})$ and $R \in \tau$.

$(a_1 \ldots a_n) \in R^{\mathcal{A}}$ iff $(h(a_1) \ldots h(a_n)) \in R^{\mathcal{B}}$

An isomorphism between $\mathcal{A}$ and $\mathcal{B}$ is a homomorphism h that is bijective.

We will say $\mathcal{A} \cong \mathcal{B}$ ($\mathcal{A}$ is isomorphic to $\mathcal{B}$) if $\exists$ isomorphism h from $\mathcal{A}$ to $\mathcal{B}$.

Proposition For every finite structure $\mathcal{A}$, $\exists$ sentence φ_A s.t.

$$\forall \mathcal{B} \models \varphi_{\mathcal{A}} \Rightarrow \mathcal{B} \cong \mathcal{A}.$$