Completeness Theorem II

Completeness Theorem Let Γ be an r.e set of sentences and φ be a sentence. The problem of determining if $\Gamma \models \varphi$ is recursively enumerable.

RE algorithm for checking logical consequence

1. Let $\Gamma' = \Gamma \cup \{\neg\varphi\}$ [$\Gamma \models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable
2. **Skolemization** Construct a set of universally quantified sentences Δ such that Γ' is satisfiable iff Δ is satisfiable. Moreover, Δ is recursively enumerable if Γ' is recursively enumerable.
2. Let

a term built for $\exists$ constant

$$\Delta_* = \{ \varphi[x_i \mapsto t_i] \mid \forall x_1 \ldots \forall x_k \varphi \in \Delta \text{ and } t_1 \ldots t_k \text{ are ground terms} \}$$

Herbrand's Theorem Δ is satisfiable iff Δ_* is satisfiable.

"Ground" Compactness Theorem Let Γ be any set of quantifier-free ground formulas. Γ is unsatisfiable iff $\exists$ a finite $\Gamma_0 \subseteq \Gamma$ such that Γ_0 is unsatisfiable.

3. Let $\Delta_0, \Delta_1 \ldots \Delta_k \ldots$ be a (computable) enumeration of finite subsets of Δ_* such that for any finite $\Delta_f \subseteq \Delta_*$, $\exists i$ s.t. $\Delta_f \subseteq \Delta_i$.

- Some finite subset of Δ
- Terms up to some finite depth
- Δ_i, set of formulas with variables replaced by bd depth terms

Algorithm

For each $\Delta_0, \Delta_1, \Delta_2 \ldots$

Decidable $\rightarrow$ Determine if Δ_i is satisfiable

If Δ_i is unsatisfiable then return "$\Gamma \models \varphi$".

Herbrand's Theorem

Herbrand Model (for signature τ)

GT_τ = Set of all ground terms of τ.

$\sim$ = congruence on GT_τ.

- $\sim$ is equivalence relation
- $\forall t_1, t_2 \ldots t_k$ and $t_1', t_2' \ldots t_k'$ s.t.

  $(t_i, t_i') \in \sim$

  then $f(t_1 \ldots t_k) \sim f(t_1' \ldots t_k')$

$A = GT_\tau \ldots$ $\mathcal{A} = (A, \{c^{\mathcal{A}}\}, \ldots \{f^{\mathcal{A}}\}, \{R\}^{\mathcal{A}} \ldots$

A structure $\mathcal{A} = (A, \{c^{\mathcal{A}}\}_{c\in\tau}, \{f^{\mathcal{A}}\}_{f\in\tau})$ rec…

is a Herbrand model if there is some

some congruence on GT_τ $\sim$ s.t.

- $A = \{ [\![t]\!]_\sim \mid t \in GT_\tau \}$

  ↳ Equivalence class of t w.r.t $\sim$

- $c^{\mathcal{A}} = [\![c]\!]_\sim$
- $f^{\mathcal{A}}([\![t_1]\!]_\sim, \ldots [\![t_k]\!]_\sim) = [\![f(t_1 \ldots t_k)]\!]_\sim$

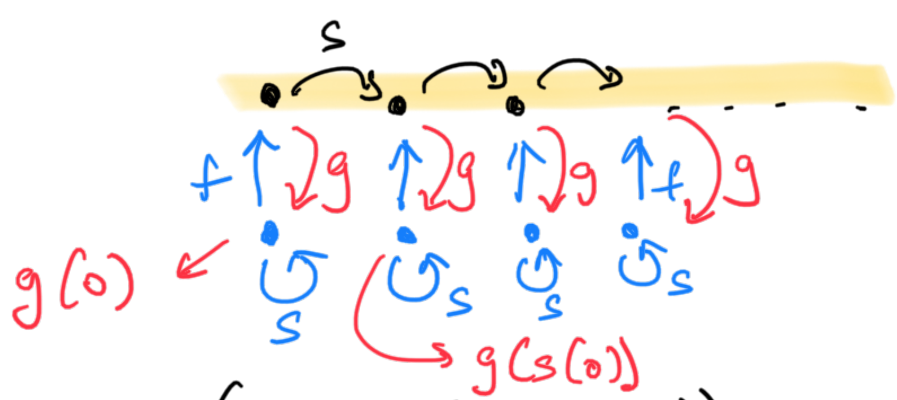
Herbrand's Theorem A universally quantified set of sentences Γ is satisfiable iff there is a Herbrand model

$$\mathcal{H} \models \Gamma.$$

Example $\tau = \{\underbrace{0}_{\text{const}}, \underbrace{s, f}_{\text{unary fns.}}\}$

$$\varphi \equiv \forall x \ \neg(s(x) = 0) \ \wedge (\forall x \forall y \ (s(x) = s(y)) \rightarrow (x = y))$$

$$\wedge (\forall x \ \exists y \ \ f(y) = x \ \wedge \ (s(y) = y))$$



$\wedge \ (\forall x \ \neg(s(x)=0)) \ \wedge \ (\forall x \forall y \ (s(x)=s(y)) \rightarrow (x=y))$

$$\psi = (\forall x\ \neg(s(x)=0)) \wedge (\forall x \forall y\ (s(x)=s(y)) \ldots)$$

$$\wedge \left(\forall x\ f(g(x)) = x \wedge (s(g(x)) = g(x))\right)$$

$$\tau' = (0, s, f, g)$$

Proof (of Herbrand's Theorem)

Let Γ is satisfiable. $\exists \mathcal{A}\ \mathcal{A} \models \Gamma$.

Define $\sim$ on GT_τ as follows

$$t_1 \sim t_2 \quad \text{iff} \quad t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$$

Interpretation of t_2 in $\mathcal{A}$

Observe $\sim$ is a congruence.

$$\mathcal{H} = \left(GT_\tau/\sim,\ \{[\![c]\!]_\sim\}_{c\in\tau},\ \{[\![f]\!]_\sim\}_{f\in\tau},\ R^{\mathcal{H}} \right\}$$

$$R^{\mathcal{H}}([\![t_1]\!]_\sim, [\![t_2]\!]_\sim \ldots [\![t_k]\!]_\sim) \text{ iff}$$

$$R^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots t_k^{\mathcal{A}})$$

$\forall\ \varphi = \forall x_1 \ldots \forall x_k\, \psi \in \Gamma$, we want to prove $\mathcal{H} \models \varphi$.

Consider assignment α for $\mathcal{H}$.

$$\alpha : x \longmapsto [\![t]\!]_\sim$$

Let $enc(\alpha)$ be an assignment for $\mathcal{A}$ defined as $enc(\alpha)(x) \mapsto t^{\mathcal{A}}$

Claim $\forall$ quantifier formula ψ and assignment α.

$\mathcal{H} \models \psi[\alpha]$ iff $\mathcal{A} \models \psi[\text{enc}(\alpha)]$

Proof By structural induction on ψ.

$\mathcal{H} \models \forall x_1 \ldots \forall x_k \psi$ iff

For every α $\mathcal{H} \models \psi[\alpha]$ iff

$\mathcal{A} \models \psi[\text{enc}(\alpha)]$

$\Uparrow$

$\mathcal{A} \models \forall x_1 \ldots \forall x_k \psi$

Corollary A set of universally quantified sentences Γ is satisfiable iff

$$\Gamma_* = \{ \psi[x_i \mapsto t_i] \mid \forall x_1 \ldots x_k \psi \in \Gamma \text{ and } t_i \text{ is a ground term} \}$$

is satisfiable.

Downward Lowenhein Skolem Theorem

Let τ be a countable signature. For any set of sentences Γ

Γ is satisfiable iff $\exists$ countable structure $\mathcal{A}$ s.t. $\mathcal{A} \models \Gamma$.

Proof $\Gamma \longrightarrow$ Skolemize Γ'

$\longrightarrow$ Herbrand model for Γ'

↳ countable because τ is countable.

Example $Th((\mathbb{R}, <))$

$\exists$ countable model $\mathcal{A}$ s.t. $\mathcal{A} \models Th(\mathbb{R}, <)$

$Th((\mathbb{R}, 0, 1, +, \times, <))$ is countable model.

Ground Compactness Theorem

Theorem Let Γ be any set of ground quantifier free formulas.

Γ is satisfiable iff every finite subset of Γ is satisfiable.

Proof Γ be a set of quantifier free ground formulas.

For every ground atomic formula a introduce a proposition p_a.

$$\Gamma_p = \{\psi[a \mapsto p_a] \mid \psi \in \Gamma\}$$

Example $\varphi = c_1 = c_2 \wedge c_2 = c_3 \wedge \neg(c_1 = c_3)$

$\varphi^p = P \quad \wedge P \quad \wedge \neg P$

$c_1 = c_2$ $\quad c_2 = c_3$ $\quad c_1 = c_3$

φ^P is satisfiable but φ is unsatisfiable.

Δ to be the set of formulas

- $\forall$ ground term t. $\quad P_{t=t}$
- $\forall$ ground terms t_1, t_2 $\quad P_{t_1=t_2} \to P_{t_2=t_1}$
- $\forall$ ground terms t_1, t_2, t_3

$$(P_{t_1=t_2} \wedge P_{t_2=t_3}) \to P_{t_1=t_3}$$

- $\forall t_1 \ldots t_k \; t_1' \ldots t_k'$

$$\left(P_{t_1=t_1'} \wedge P_{t_2=t_2'} \ldots \wedge P_{t_k=t_k'}\right) \to P_{f(t_1 \ldots t_k)=f(t_1' \ldots t_k')}$$

- $\forall t_1 \ldots t_k \; t_1' \ldots t_k'$

$$\left(P_{t_1=t_1'} \wedge \ldots \wedge P_{t_k=t_k'} \wedge P_{R(t_1 \ldots t_k)}\right) \to P_{R(t_1' \ldots t_k')}$$

Any subset of $\Delta \cup \Gamma_p$. is satisfiable iff corresponding subset of Γ is satisfiable

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$\Delta \cup \Gamma_p$ is satisfiable iff

$\forall$ finite subset $\Gamma_0 \subseteq \Delta \cup \Gamma_\phi$ is satisfiable.