# Completeness Theorem

Define a proof system.

$\Gamma \vdash \varphi$

Completeness Theorem $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$

If $\Gamma \models \varphi$ then there exists a proof of $\varphi$ from $\Gamma$

— Typically you mechanical check if a proof is correct

Completeness Theorem Given recursively enumerable $\{\Gamma$ and $\varphi,\}$ the problem of determining if $\Gamma \models \varphi$ is recursively enumerable.

Proof Outline

$\Gamma \models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable

1. For any set of sentences $\Gamma$ we can construct a set $\Delta$ of universally quantified sentences such that

   $\forall x_1 \forall x_2 \ldots \forall x_k (\psi)$ → quantifier free

   $\Gamma$ is satisfiable iff $\Delta$ is satisfiable

2. Herbrand's Theorem

Ground Term   Every constant in $C$ is term

~~Every variable is a term~~

For terms $t_1 \ldots t_k$ and $k$-ary function symbol $f$,

$f(t_1 \ldots t_k)$ is term.

Ground Formulas

If $t_1, t_2$ are ground terms then

$t_1 = t_2$ is (atomic) formula

If $t_1 \ldots t_k$ are ground terms & $R$ is $k$-ary relation then $R(t_1 \ldots t_k)$ is fml

If $\varphi$ & $\psi$ are formulas then

$\varphi \vee \psi$ is formula

If $\varphi$ is formula then $\neg\varphi$ is wff

If $\varphi$ is formula & $x$ is variable

then $\exists x \varphi$ is wff.

Let $\Gamma$ be a set of universally quantified sentences.

$$\Gamma^* = \{ \varphi[x_i \mapsto t_i] \mid \forall x_1 \ldots \forall x_k \underbrace{\varphi}_{\text{quantifier-free}} \in \Gamma, \; t_1 \ldots t_k \text{ are ground terms} \}$$

Set of quantifier free ground formulas.

$\Gamma$ is satisfiable iff $\Gamma^*$ is satisfiable.
(a set of qf ground wffs)

3. "Ground" Compactness Theorem $\Gamma$ is unsatisfiable iff $\exists$ a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0$ is unsatisfiable.

4. Given a finite quantifier-free set of formulas $\Gamma$, determining if $\Gamma$ is satisfiable is decidable. $\leftarrow$ Proved on October 7.

Algorithm for determining $\Gamma \models \varphi$. ($\Gamma$ is r.e)

<u>Definition of R.E</u>: L is r.e. if there is a TM that outputs all the strings in L

If $\Gamma$ is r.e then the equisatisfiable set of universal sentences $\Delta$ is also r.e. (Assume $\mathcal{C}$ is finite)

1. $\Sigma = \phi$, $d = 0$
2. Repeat forever
3. Enumerate the next sentence in $\Delta$ $\varphi$. $\Sigma = \Sigma \cup \{\varphi\}$
4. $d{+}{+}$
5. $T$ = Set of all ground terms of depth $\leq d$.

6. Construct the set $\Sigma^*$ obtained by substituting every variable by some term in T in formulas in $\Sigma$.

7. Determine if $\Sigma^*$ is satisfiable

8. If $\Sigma^*$ is unsatisfiable then return unsat

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If $\Gamma$ is unsat $\Rightarrow$ $\Delta$ is unsat

$\Rightarrow$ $\exists$ finite subset of $\Delta$ instantiated with ground terms that is unsat

$\Rightarrow$ $\exists$ same set $\Sigma^*$ in my enumeration that is unsat.

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Constructing an equisatisfiable set of universally quantified sentences.

Consider a sentence

Not in prenex

$$\forall x \left( (\exists x\, P(x)) \vee \neg \left( \neg (\exists x\, Q x) \vee R(x) \right) \right)$$

Prenex Normal Form A formula $\varphi$ is in prenex normal form if it is of the form $Q_1 x_1 Q_2 x_2 \ldots Q_k x_k\ \psi$ where each $Q_i \in \{\exists, \forall\}$ and $\varphi$ is quantifier free

$$\forall x \exists y \exists z \left( P(y) \vee (Q(z) \wedge R(x)) \right)$$

prenex.

## Converting to prenex normal form

1. Push negations to atomic formulas

$$\forall x \; (\exists x \; P(x)) \vee \neg(\neg(\exists x \; Q(x)) \vee R(x))$$

$$\equiv \forall x \left( (\exists x \; P(x)) \vee ((\exists x \; Q(x)) \wedge \neg R(x)) \right)$$

2. Rename bound variable to ensure that each bound variable is distinct

$$\forall x \left( (\exists x \; P(x)) \vee ((\exists x \; Q(x)) \wedge \neg R(x)) \right)$$

$$\equiv \forall x \left( (\exists y \; P(y)) \vee ((\exists z \; Q(z))) \wedge \neg R(x) \right)$$

3. Pull all the quantifiers to the front maintaining the same order.

$$\forall x \; (\exists y \; P(y)) \vee ((\exists z \; Q(z)) \wedge \neg R(x))$$

$$\equiv \forall x \exists y \exists z \quad P(y) \vee (Q(z) \wedge \neg R(x))$$

## Skolemization

Consider $\overbrace{\forall x_1 \forall x_2 \ldots \forall x_k \exists y \; \varphi(x_1, \ldots x_k, y)}^{\text{over the signature } \Sigma}$

For every value given to $x_1 \ldots x_k$, there is a value for $y$ s.t $\varphi$ holds.

Consider new formula $\tau \cup \{f\}$ $f \notin \tau$.

$$\forall x_1 \forall x_2 \ldots \forall x_k \; \psi \, [y \longmapsto f(x_1 \ldots x_k)]$$

$\hookrightarrow$ no $y$.

Claim $\forall x_1 \ldots \forall x_k \exists y \, \psi$ is sat iff

$\forall x_1 \ldots \forall x_k \; \psi \, [y \mapsto f(x_1 \ldots x_k)]$ is sat.

$$\mathcal{A} \models \underline{\forall x_1 \ldots \forall x_k \exists y \; \psi [\alpha]}$$

$\mathcal{A}' = (\mathcal{A}, f :$ corresponds to sat formula$)$

$$\forall x \exists y \exists z \, (P(y) \vee (Q(z) \wedge \neg R(x)))$$

$$\forall x \; P(g(x)) \vee (Q(h(z)) \wedge \neg R(x))$$