Quantifier free theory of equality

Determining $\varphi \in Th(\mathcal{A})$. This problem is decidable

- $\mathcal{A} = (\mathbb{R}, <)$
- $\mathcal{A} = (\mathbb{R}, 0, 1, +, <)$

$\Big\}$ Quantifier eliminations

Determine if $\models \varphi$

- Problem is RE-hard.
- Problem is in RE (Gödel's completeness)

Given a sentence φ of the form $\forall x_1 \forall x_2 \ldots \forall x_k \psi$ where ψ is quantifier-free

Determine if φ is valid.

Show This problem is decidable

Definition A formula φ is valid iff $\forall$ structures $\mathcal{A}$ and assignments α

$$\mathcal{A} \models \varphi[\alpha].$$

ψ is valid iff sentence $\forall x_1 \ldots \forall x_k \psi$ is valid ($\mathrm{free}(\psi) \subseteq \{x_1 \ldots x_k\}$)

This problem is refered to determining the quantifier-free theory of equality.

Signatures $(\mathcal{C}, \mathcal{F}, \mathcal{R})$

Constants Function Relations

Terms
- Variable
- Constant symbol
- If f is k-ary function, and $t_1, \ldots, t_k$ are terms then $f(t_1, \ldots, t_k)$ term.

Example $\tau = (0, 1, +, <)$ ($+$ is binary fn)

$+(0, +(1, +(1, x)))$ — term

$x + \boxed{1+1}$ — terms

Atomic fmls.

Formulas
- $t_1 = t_2$ (t_1, t_2 terms)
- $R(t_1, \ldots, t_k)$ (t_i term)
- $\neg \psi$
- $\varphi \vee \psi$
- $\exists x\ \varphi$.

Structure $\mathcal{A} = (A, c^{\mathcal{A}}, \{f^{\mathcal{A}}\}_{f \in \tau}, \{R^{\mathcal{A}}\}_{R \in \tau})$
- $c^{\mathcal{A}} \in A$
- $R^{\mathcal{A}} \subseteq A \times A \times \ldots A$
- $f^{\mathcal{A}} : A \times A \times \ldots A \to A$.

Today: WLOG assume τ only consists of constants and function symbols. There are no relation symbols.

If $R \in \tau$, τ' where add Π and function symbol f_R

- Replace $R(t_1 \dots t_k) \mapsto f_R(t_1 \dots t_k) = \Pi$

We quantifier free formula φ and we want determine if φ is valid.

- Every atomic formula $t_1 = t_2$.

Goal Give a decision procedure for the following problem: Given a quantifier-free formula φ determine if φ is satisfiable.

i.e. $\exists \mathcal{A}$ and α s.t.

$$\mathcal{A} \models \varphi[\alpha]$$

Theorem A quantifier free formula φ is satisfiable iff $\exists \mathcal{A}$ and α s.t. $|u(\mathcal{A})| \le |\varphi|$ and $\mathcal{A} \models \varphi[\alpha]$.

Proof ($\Leftarrow$) Obvious

($\Rightarrow$) Assume that $\mathcal{A} \models \varphi[\alpha]$

$S = \text{terms}(\varphi)$

$A' = \{ \alpha(t) \mid t \in S \}$

↪ Value of t in $\mathcal{A}$ w.r.t α.

$|A'| \le |S| \le |\varphi|$

$|A'| = |S| = \ldots$

Assume WLOG $A' \neq \emptyset$

- if terms $(\psi) = \emptyset$ then terms $(\psi \wedge (x = x))$

Consider $\mathcal{A}' = (A', \{c^{\mathcal{A}'}\}, \{f^{\mathcal{A}'}\})$ where

$$c^{\mathcal{A}'} = \begin{cases} c^{\mathcal{A}} & c \in S \\ * & c \notin S \text{ where } * \in A'. \end{cases}$$

$$f^{\mathcal{A}'}(e_1 \ldots e_k) = \begin{cases} \alpha(f(t_1 \ldots t_k)) & f(t_1 \ldots t_k) \in S \\ * & f(t_1 \ldots t_k) \notin S \end{cases}$$

each $e_i \in A'$

$\Rightarrow \exists t_i \in S$ s.t $\alpha(t_i) = e_i$

$$\alpha'(x) = \begin{cases} \alpha(x) & x \in S \\ * & x \notin S \end{cases}$$

Inductively, for every $t \in S$.

$$\alpha'(t) = \alpha(t)$$

Inductively

$$\mathcal{A} \models \psi[\alpha] \text{ iff } \mathcal{A}' \models \psi[\alpha']$$

NP-algorithm to check if ψ is satisfiable

- Guess a universe
- Guess the value each term in S takes
- Evaluate ψ in this structure.

Proposition Determining if a quantifier-free

...

formula ψ is satisfiable is NP-complete.

Proof NP-hard because Sat is NP-hard for propositional logic.

— Reduction. Each proposition p is variable p. Add constant $\top$

$$p \mapsto (p = \top)$$

Conjunctive Formulas

Consider ψ of the form $\alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_k$ where each α_i is literal { Either atomic or negation of atomic.

Atomic $t_1 = t_2$

So ψ of the form $\alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_k$ where α_i is either $t_1 = t_2$
or $\neg(t_1 = t_2)$ $[t_1 \neq t_2]$

Given $\psi = \alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_k$ define

$$E = \{(t_1, t_2) \mid \exists \alpha_i = (t_1 = t_2)\}$$

$$D = \{(t_1, t_2) \mid \exists \alpha_i = (t_1 \neq t_2)\}$$

Equality :

$$t = t \qquad \text{(Reflexivity)}$$

$$t_1 = t_2 \Rightarrow t_2 = t_1 \qquad \text{(Symmetry)}$$

$$t_1 = t_2 \wedge t_2 = t_3 \Rightarrow t_1 = t_3 \qquad \text{(Transitive)}$$

$t_1 = t_1' \wedge \dots \; t_k = t_k' \Rightarrow f(t_1 \dots t_k) = f'(t_1' \dots t_k')$

(Cong)

Congruence Closure (E) : C smallest relation $\subseteq term(\psi) \times term(\psi)$

- $E \subseteq C$
- $\forall t \in terms(\psi) \,.\, (t, t) \in C$
- $\forall t_1, t_2 \; (t_1, t_2) \in C \Rightarrow (t_2, t_1) \in C$
- $\forall t_1, t_2, t_3 \; (t_1, t_2) \in C, (t_2, t_3) \in C \Rightarrow (t_1, t_3) \in C$.
- $(t_1, t_1') \in C \dots (t_k, t_k') \in C \Rightarrow (f(t_1 \dots t_k), f(t_1' \dots t_k')) \in C$.

$C_0 = E$

$C_{i+1} = C_i \cup \dots$

$terms(\psi) \leq |\psi| = n$

$|C| \leq n^2$

Proposition ψ is satisfiable iff Congruence Closure $(E) \cap D = \emptyset$

Proof If $\mathcal{A} \models \psi[\alpha]$

$\forall (t_1, t_2) \in Cong\,Cl(E)$.

$\alpha(t_1) = \alpha(t_2)$

$\forall (t_1, t_2) \in D \quad \alpha(t_1) \neq \alpha(t_2)$

$\forall (t_1, t_2) \in \mathcal{D}, \quad \ldots$

$(\Leftarrow)$ $\mathrm{CongCl}(E) \cap D = \emptyset$

$$A = \mathrm{terms}(\Psi) / \mathrm{CongCl}(E)$$

$$c^{A} = \begin{cases} [c] & \text{if } c \in \mathrm{Terms}(\Psi) \\ * & \text{o.w} \end{cases}$$

$*$ some $\in A$.

$$f^{A}(\quad \ldots) = \begin{cases} [f(\ldots)] & \text{if } f(t \ldots) \in \Psi \\ * & \text{o.w} \end{cases}$$