Church-Turing Theorem

The Classical Decision Problem Given a sentence φ over signature τ, determine if $\models \varphi$, i.e. $\forall$ τ-structures $\mathcal{A}$, $\mathcal{A} \models \varphi$.

Church-Turing Theorem Given a sentence φ the problem of determining if φ is valid is RE-hard.

Corollary Checking validity is undecidable.

Trakhtenbrot's Theorem Given a sentence φ the problem of determining if φ is true in all <u>finite</u> structures is coRE hard.

Church-Turing Theorem

$MP = \{ \langle M, w \rangle \mid w \in L(M) \}$

MP is RE-complete.

There is TM U s.t. $L(U) = MP$.

Proof outline: $MP \leq_m \text{Valid}$

Given α we effectively construct a sentence φ_α s.t. $\alpha \in MP$ iff φ_α is valid.

φ_α will depend on U and α

- Encodes conditions that say U has an accepting computation on α.

Signature of $\varphi_\alpha = \{0, s, St@, InpSym, InpHd, TapeSym, TapeHd\}$

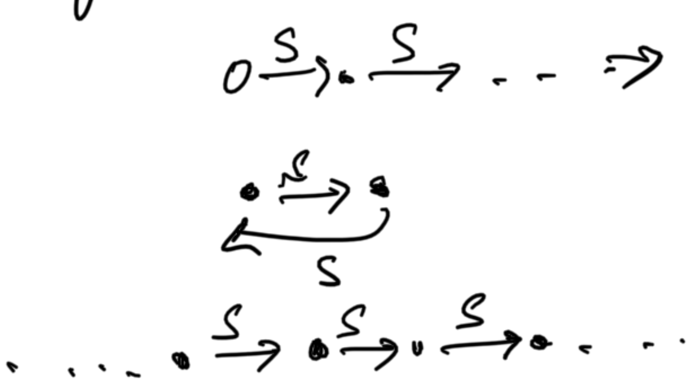
Serial $= \forall x \exists y \; s(x,y)$

Right-unique $= \forall x \forall y \forall z \, (S(x,y) \wedge S(x,z)) \rightarrow y = z$

Injective $= \forall x \forall y \forall z \, (S(x,z) \wedge S(y,z)) \rightarrow (x = y)$

Zero $= \forall x \, \neg S(x, 0)$

If $\mathcal{A} \models$ Serial $\wedge$ Right Unique $\wedge$ Injective $\wedge$ Zero then there is some subset of $\mathcal{A}$ that is isomorphic to $\mathbb{N}$.



$\varphi_{\mathbb{N}} =$ Serial $\wedge$ Right Unique $\wedge$ Injective $\wedge$ Zero

Assumptions about U WLOG.

- U has one worktape and one input tape
- U has n states, q_0 initial state, q_{acc}. States encoded as $\{0, 1 \ldots n-1\}$
- U has tape alphabet Γ containing
  1) $\triangleright$ (left end marker). The tape

alphabet is encoded as $\{0, \ldots m-1\}$

Intuition behind relation symbols

$St@(q,t)$ — if state of U is q at time t

$InpSym(a,c)$ — if symbol a is in cell c.

$InpHd(c,t)$ — if input head is on cell c at time t

$TapeSym(a,c,t)$ — The symbol in cell c at time t is a on work-tape

$TapeHd(c,t)$ — Work tape head at time t is on cell c.

Goal Write down the property that U has an accepting computation on α.

Auxiliary Formulas

"Variable x stores a value which number k"

$$k(x) = \exists x_1 \exists x_2 \cdots \exists x_k \; s(0,x_1) \wedge s(x_1,x_2) \ldots s(x_{k-1},x_k) \wedge x = x_k.$$

"Argument to R is the symbol a which encoded by number k"

$$R(\underline{a}, \ldots) = \exists x \; R(x, \ldots) \wedge k(x)$$

"x stores a value which is state"

$$State(x) = (x=0) \vee \bigvee_{k=1}^{n-1} k(x)$$

"x stores a value which is tape symbol"

$$\text{Symbol}(x) = (x=0) \vee \bigvee_{k=1}^{m-1} k(x)$$

$$\varphi_{const} = \forall t\, \forall x\, \forall y\, \big(St@(x,t) \wedge St@(y,t)\big) \rightarrow x = y$$

$\wedge\ \forall t$ Tape cell hold unique value

$\wedge\ \forall t$ Head position is unique

$$\varphi_{init} = St@(\underline{q_0}, 0) \wedge \text{TapeSym}(\underline{\triangleright}, 0, 0) \wedge \big(\forall c\ \neg(c=0) \rightarrow \text{Tape}(\underline{\sqcup}, c, 0)\big)$$

$$\wedge\ \text{TapeHd}(0,0) \wedge \text{InpHd}(0,0) \wedge \text{InpSym}(\underline{\triangleright}, 0) \wedge \bigwedge_{k=1}^{\ell} \text{InpSym}(\underline{\alpha[k]}, \underline{k})$$

$\wedge$ InpSym is $\underline{\sqcup}$ everywhere else

$$\varphi_{move} = \bigwedge_{\delta(q,i,a) = (q', d_i, b, d_t)} \varphi_{q,i,a \rightarrow q', d_i, b, d_t}$$

$d_i = -1, \quad d_t = -1$

$$\varphi_{q,i,a \rightarrow q', d_i, b, d_t} = \forall t\, \forall t'\ \exists c_i \exists c_t$$

$$S(t, t') \wedge St@(q, t) \wedge$$

$$\text{InpSym}(\underline{i}, c_i) \wedge \text{InpHd}(c_i, t) \wedge$$

$$\text{TapeSym}(\underline{a}, c_t, t) \wedge \text{TapeHd}(c_t, t) \wedge$$

$\rightarrow \mathrm{TapeSym}(\underline{b}, c_t, t) \wedge$

$(\forall x \forall c \neg(c = c_t) \rightarrow (\mathrm{TapeSym}(x, c, t) \leftrightarrow \mathrm{TapeSym}(x, c, t')))$

: Head moves in the right direction.

$$\varphi_{accept} = \exists t \ St@(\underline{q_a}, t)$$

$$\varphi_\alpha = (\varphi_{\mathbb{N}} \wedge \varphi_{const} \wedge \varphi_{init} \wedge \varphi_{move}) \rightarrow \varphi_{accept}.$$

If φ_α is valid. Then consider $\mathcal{A}$ where Universe is $\mathbb{N}$, and the predicates $St@, \ldots$ are interpreted to be consistent with the computation of U on α.

$\mathcal{A} \models \varphi_{\mathbb{N}} \wedge \ldots \quad \varphi_{move}$

$\mathcal{A} \models \varphi_{accept}$

If U accepts α.

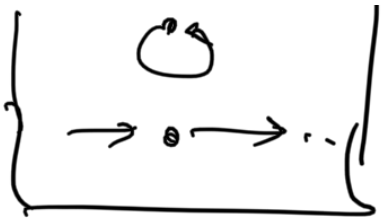
Consider some structure $\mathcal{A}$

$\mathcal{A} \models \varphi_{\mathbb{N}} \wedge \varphi_{const} \wedge \varphi_{init} \wedge \varphi_{move}$

Since $\mathcal{A} \models \varphi_{\mathbb{N}}$

$$\underline{\underline{A'}} \begin{cases} 0 \rightarrow 0 \rightarrow \ldots \\ 0 \end{cases}$$

Interpretation of predicates restricted to A', that will correspond to a computation of U on α. (Induction)



Since U accepts α, on the universe A' there will be some time where $St@$ corresponds to U being in the accept state. $\mathcal{A} \models \varphi_{accept}$.

$$\varphi'_\alpha = \left(\varphi_{IN} \wedge \varphi_{init} \dots \wedge \varphi_{move}\right) \to \varphi_{\substack{Not \\ accept}}$$

$$\varphi_{Not\ Accept} = \forall t\ \neg St@(\underline{q_{a}}, t)$$

Even if U does not accept α, $\cancel{\models}$ φ'_α need not be valid.