Quantifier Elimination: Linear Arithmetic

Definition A theory T admits quantifier elimination if for every formula $\varphi(\vec{x})$, there is quantifier-free formula $\varphi'(\vec{x})$ such that

$$T \models (\varphi \to \varphi') \land (\varphi' \to \varphi)$$

Proposition If every formula of the form $\exists x\, \alpha_1 \land \alpha_2 \ldots \land \alpha_k$ (where α_i is a literal) is equivalent to a quantifier-free formula with respect to theory T, then T admits quantifier elimination.

atomic / not atomic

Linear Arithmetic

$Th((\mathbb{R}, 0, 1, +, <))$ admits quantifier elimination.

Ternary relation: $+(x, y, z)$ if z is the sum of x and y.

$$\forall x\, (0 < x \to 1 < x + y)$$

$$\equiv \forall x \forall z\, (0 < x \to (+(x, y, z) \to 1 < z))$$

Fourier - Motzkin

$\exists x\, \alpha_1 \land \alpha_2 \ldots \land \alpha_k$ where α_i is literal

$\exists x\ \alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_k$ where α_i is literal.

Atomic $s < t$, $s = t$, s, t are expressions

$$\neg(s<t) \equiv (s=t) \vee (t<s)$$

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Goal Eliminate quantifiers from

$\exists x\ \beta_1 \wedge \beta_2 \ldots \beta_k$ where β_i is atomic

Step 1 "Solve for x"

Transform each β_i into $x < t$

$x = t$

or $t < x$

where x does not appear in t.

$t = \sum a_i y_i$ where a_i are rational numbers.

Example $\frac{1}{2}x + 3y < 2x + \frac{1}{3}y$

$$3x + 9y < 4x + 2y$$

$$x + x + x + y + \cdots y < x + x + x + x + y + y$$

$$\exists x \bigwedge_{l \in L} l < x \wedge \bigwedge_{e \in E} x = e \wedge \bigwedge_{u \in U} x < u$$

Step 2

Case 1 : $E \neq \emptyset$

$$\exists x\ x = t \wedge \psi \equiv \psi[x \mapsto t]$$

Case 2 : $E = \emptyset$

$$\exists x \bigwedge_{l \in L} l < x \wedge \bigwedge_{u \in U} x < u \equiv \bigwedge_{l \in L, u \in U} l < u .$$

$\exists x\, \varphi(x, \vec{y})$

- Transform $\bigvee \exists x\, \beta_1 \wedge .. \beta_k$ where β_i is atomic.

  Involves converting into DNF

  Can we avoid?

Ferrante - Rackoff

$\exists x\, \varphi(x, \vec{y})$

- Push negations inside in φ resulting in

  $\exists x\, \psi(x, \vec{y})$ where

  ψ is a positive Boolean combination of literals.

- Eliminating negation we get.

  $\exists x\, \eta(x, \vec{y})$ where

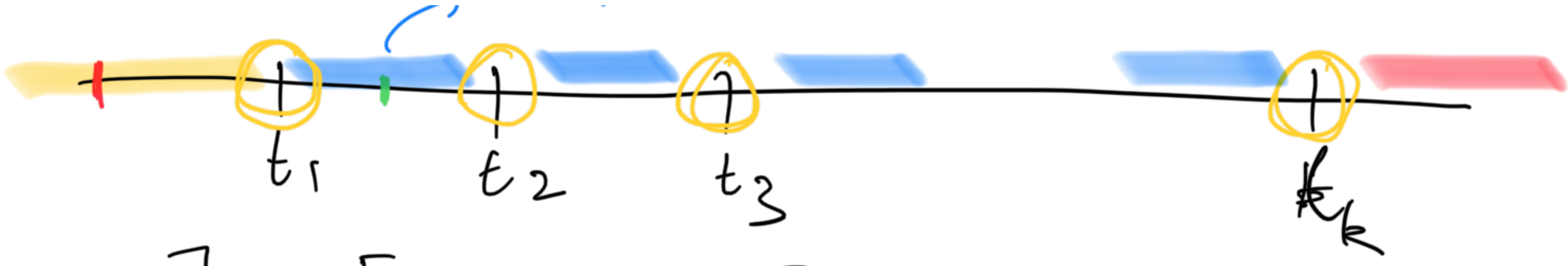
  η is positive Boolean comb. of atomic fmls.

- For each atomic formula, we solve for x.

  $\exists x$ +ve Boolean comb $(x < t,\ x = t,\ t < x)$

Imagine values are assigned to all variables other than x.

→ midpoint



$\exists x\ F$ where F is a +ve Boolean combination of constraints $x < t$, $x = t$, $t < x$.

$$F_{-\infty} \equiv F[x < t \mapsto \top,\ x = t \mapsto \bot,\ t < x \mapsto \bot]$$

$$F_{+\infty} \equiv F[x < t \mapsto \bot,\ x = t \mapsto \bot,\ t < x \mapsto \top]$$

For $x \bowtie_1 t_1,\ x \bowtie_2 t_2 \in F$ $[\bowtie_1, \bowtie_2 \in \{<, =, >\}]$

$$F_{t_1, t_2} \equiv F\left[x \mapsto \frac{t_1 + t_2}{2}\right]$$

$$\exists x F \equiv F_{-\infty} \vee F_{+\infty} \bigvee_{\substack{x \bowtie_1 t_1 \\ x \bowtie_2 t_2 \\ \in F}} F_{t_1, t_2}$$

Example $\forall x\ \ 0 < x \to 1 < x + y$

Quantifier free equivalent formula is $1 < y \vee 1 = y$

$$\varphi \equiv \neg \exists x\ \ 0 < x \wedge \neg(1 < x + y)$$

$$\equiv \neg \exists x\ \ 0 < x \wedge (1 = x + y \vee x + y < 1)$$

$\exists x\ \psi$.

$$\exists x \psi \equiv \exists x\ \ \underbrace{0 < x \wedge (x = 1 - y \vee x < 1 - y)}_{F}$$

$$F_{-\infty} \equiv \bot \wedge (\bot \vee \top) \equiv \bot$$

$$F_{+\infty} \equiv \top \wedge (\bot \vee \bot) \equiv \bot$$

$\underline{t_1 + t_2}$: $0,\ 1 - y,\ \underline{1 - y}$.

$$F_0 \equiv 0<0 \wedge (0 = 1-y \vee 0 < 1-y) \equiv \bot$$

$$F_{1-y} \equiv 0 < 1-y \wedge (1-y = 1-y \vee 1-y < 1-y) \equiv y<1$$

$$F_{\frac{1-y}{2}} \equiv 0 < \frac{1-y}{2} \wedge \left(\frac{1-y}{2} = 1-y \vee \frac{1-y}{2} < 1-y\right)$$

$$\equiv y<1 \wedge (1-y = 1-y+1-y \vee 1-y < 1-y+1-y)$$

$$\equiv y<1 \wedge (0 = 1-y \vee 0 < 1-y) \equiv y<1$$

$$\exists x \psi \equiv F_{-\infty} \vee F_{+\infty} \vee F_0 \vee F_{1-y} \vee F_{\frac{1-y}{2}}$$

$$\equiv \bot \vee \bot \vee \bot \vee y<1 \vee y<1 \equiv y<1$$

$$\varphi \equiv \neg \exists x \psi$$

$$\equiv \neg(y<1) \equiv y=1 \vee 1<y.$$

$Th((\mathbb{R}, 0, 1, +, <))$ admits quantifier elimination.

Corollary $Th(\mathbb{R}, 0, 1, +, <)$ is decidable

$$Th(\mathbb{Q}, 0, 1, +, <) = Th(\mathbb{R}, 0, 1, +, <)$$

Tarski-Seidenberg Theorem

$Th(\mathbb{R}, 0, 1, +, \times, <)$ admits quantifier elimination and is decidable.

$Th(\mathbb{C}, 0, 1, +, \times, <)$ admits quantifier elimination and is decidable.

Robinson $Th(\mathbb{Q}, 0, 1, +, \times, <)$ is not recursively enumerable.

Gödel $Th(\mathbb{N}, 0, 1, +, \times, <)$ is not recursively enumerable.

Presburger $Th(\mathbb{N}, 0, 1, +, <)$ is decidable

Skolem $Th(\mathbb{N}, 0, 1, \times, <)$ is decidable