Quantifier Elimination: Dense Linear Orders

truth assign (prop) / structure (FO)

Model Checking Given a model M and a formula φ, determine if

$$M \models \varphi$$

- Model checking for FO when M is finite this decidable
- Model checking FO when M is infinite decidability is unclear.

Need a computable representation of M when M is infinite.

- Given a constant c, interpretation c^M should be computable.
- Given any R, and $\vec{x}$ (vector of values), it should be decidable to determine if $R^M \vec{x}$

* $u(M)$ — recursive / decidable. Not required

Quantifier Elimination

Theory is just a set of sentences.

Definition A theory T admits quantifier elimination if for every formula $\varphi(\vec{x})$ - there is a quantifier-free formula

$\varphi'(\vec{x})$ s.t. $\longrightarrow$ Formula that does not use $\exists/\forall$

$$T \models (\varphi \to \varphi') \wedge (\varphi' \to \varphi)$$

If the construction of the quantifier-free formula φ' is computable and if the problem $T \models \varphi$ is decidable when φ is quantifier-free then the problem of determining if $T \models \varphi$ is decidable.

$$\varphi ::= t = t \mid Rt \ldots t \mid (\neg \varphi) \mid (\varphi \vee \varphi) \mid (\exists x\, \varphi)$$

either constant or variable

Atomic formulas $t = t$, $Rt \ldots t$

Literal Either an atomic formula or its negation.

Proposition Theory T. If every formula of the form $(\exists x\, \alpha_1 \wedge \alpha_2 \ldots \alpha_n)$ (where α_i is a literal) is equivalent to a quantifier-free formula (w.r.t T) then every formula is equivalent to a quantifier free formula.

Proof: For a formula ψ, $qf(\psi)$ to be the quantifier-free formula constructed in this proof.

Base case Trivial

Induction step.

- $\neg\psi$. $\quad qf(\neg\psi) = \neg qf(\psi)$
- $\psi_1 \vee \psi_2 \quad qf(\psi_1 \vee \psi_2) = qf(\psi_1) \vee qf(\psi_2)$
- $\exists x\, \psi. \equiv \underbrace{\exists x.\, qf(\psi)}$

$qf(\psi)$ — may not be a conjunction of literals.

Disjunctive Normal Form

— Disjunction of conjunctions of literals.

$$DNF(qf(\psi)) \equiv \tau_1 \vee \tau_2 \vee \dots \tau_k$$

$$\exists x\, qf(\psi) \equiv (\exists x\ \tau_1 \vee \tau_2 \dots \tau_k)$$

$$\equiv \exists x\, \tau_1 \vee \exists x\, \tau_2 \dots \vee \exists x\, \tau_k$$

$$\equiv \bigvee_{i=1}^{k} qf(\exists x\, \tau_i)$$

Theorem The problem of determining if a

sentence φ holds in $(\mathbb{R}, <)$ is decidable.

Theorem $Th(\mathbb{R}, <)$ admits quantifier elimination.

$(\mathbb{R}, <)$ satisfies the following sentences

(IRref) $\forall x \ \neg(x < x)$

(ASym) $\forall x \forall y \ (x < y) \to \neg(y < x)$

(Trans) $\forall x \forall y \forall z \ (x < y \wedge y < z) \to x < z$

(Total) $\forall x \forall y \ \ x < y \vee y < x \vee x = y$

(Dense) $\forall x \forall y \ \ x < y \to (\exists z \ \ x < z \wedge z < y)$

(No Small) $\forall x \ \exists z \ \ z < x$

(No Large) $\forall x \ \exists z \ \ x < z$

Dense linear order without endpts (DLOWE)

Proposition Every formula of the form $\exists x \ \alpha_1 \wedge \alpha_2 \ldots \alpha_k$ where α_i is a literal, is equivalent to a formula of the form

$$\bigvee_{j=1}^{m} \left(\exists x \ \beta_1 \wedge \beta_2 \wedge \ldots \beta_{k_j}\right)$$

where each β is an atomic formula.

Proof Atomic formulas are of the form

$x = y, \quad x < y$

$\neg(x=y) \equiv x<y \vee y<x.$ (totality)

$\neg(x<y) \equiv x=y \vee y<x$ (totality)

Convert each α_i into a formula that does not use negations.

Convert $\bigwedge \alpha_i$ in DNF

Push $\exists x$ inside

Need to Eliminate quantifiers from formulas of the form $\exists x\ \beta_1 \wedge \cdots \wedge \beta_k$ where each β_i is atomic formula.

WLOG: No β_i is $x = x$

No β_i is $y \bowtie z$ where $y, z \neq x$

(Because $\exists x\,(y \bowtie z \wedge \beta_1 \cdots \beta_{k-1})$

$\equiv y \bowtie z \wedge \exists x\, \beta_1 \cdots \beta_{k-1}$)

Considering $\exists x\ \beta_1 \wedge \cdots \beta_k$ where

- each β_i is atomic
- x appears in each β_i.
- $x = x$ is not one of the β_i's.

Case 1 $\exists x\ x = t \wedge \varphi$

$\equiv \varphi[x/t]$

Case 2 $\exists x \bigwedge_{y \in U} x < y \wedge \bigwedge_{z \in L} z < x$

$$\equiv \bigwedge_{z \in L, y \in U} z < y.$$

if empty $\top$

Decision procedure for $(\mathbb{R}, <) \models \varphi$ and φ is sentence

- Convert φ ~~to~~ a quantifier free formula ψ
- Check if $\psi \equiv \top$.

Let T be the sentences defining DLOWE

There is decision procedure to decide if $T \models \varphi$

$(\mathbb{Q}, \leq) \models$ DLOWE