First order logic

Predicates Propositions whose truth depends values given to certain parameters

"x is prime"

"$x + y = 0$"

Predicates $\longrightarrow$ Proposition

Fix values to parameters

$\exists x$ x is prime — quantifiers.

Syntax

Signature / Vocabulary $\{\mathcal{C}, \mathcal{R}\}$

$\mathcal{C}$ is set of constants

$\mathcal{R} = \{R_i^k\}$ where R_i^k is k-ary relation symbol.

Finite signature: the $\mathcal{C}$ and $\mathcal{R}$ are finite.

Variables Set of variables $\mathcal{V}$.

Well formed Formulas

$$\varphi ::= t = t \mid Rtt\ldots t \mid (\neg\varphi) \mid (\varphi \vee \varphi) \mid (\exists x\, \varphi)$$

t — either a constant symbol or variable

$Rtt\ldots t$ — # arguments match arity of R

x — variable

Semantics

$\mathcal{A} = (A, \{c^{\mathcal{A}}\}, \{R^{\mathcal{A}}\})$

Structure $\mathcal{A} = (A, \{c\}_{c \in \mathcal{C}}, \{R\}_{R \in \mathcal{R}})$

A – universe (set)

$c^{\mathcal{A}}$ – interpretation of constant c in $\mathcal{A}$

$c^{\mathcal{A}} \in A$

$R^{k,\mathcal{A}}$ – interpretation of relation R in $\mathcal{A}$

$$R^{k,\mathcal{A}} \subseteq (\underbrace{A \times A \times \cdots A}_{k})$$

Assignment $\alpha : \mathcal{V} \to u(\mathcal{A})$

$\alpha(v)$ – value given to variable v in assignment α

$\alpha(c) = c^{\mathcal{A}}$ for constants c.

Given assignment α, variable x and $a \in u(\mathcal{A})$ the $\alpha[x \mapsto a]$ is given by

$$\alpha[x \mapsto a](y) = \begin{cases} \alpha(y) & \text{if } y \neq x \\ a & \text{if } y = x \end{cases}$$

Satisfiability $\mathcal{A} \models \varphi[\alpha]$

"φ is true in $\mathcal{A}$ w.r.t α".

$\mathcal{A} \models t_1 = t_2\,[\alpha]$ iff $\alpha(t_1) = \alpha(t_2)$

$\mathcal{A} \models R^k t_1 \ldots t_k\,[\alpha]$ iff $(\alpha(t_1) \ldots \alpha(t_k)) \in R^{k,\mathcal{A}}$

$\mathcal{A} \models (\neg \varphi)[\alpha]$ iff $\mathcal{A} \not\models \varphi[\alpha]$

$\mathcal{A} \models (\varphi \vee \psi)[\alpha]$ iff $\mathcal{A} \models \varphi[\alpha]$ or $\mathcal{A} \models \psi[\alpha]$.

$\mathcal{A} \models (\exists x\ \varphi)[\alpha]$ iff $\mathcal{A} \models \varphi[\alpha[x \mapsto a]]$ for

... iff ... $[\alpha[x \mapsto a]]$ for some $a \in \underline{u(\mathcal{A})}$

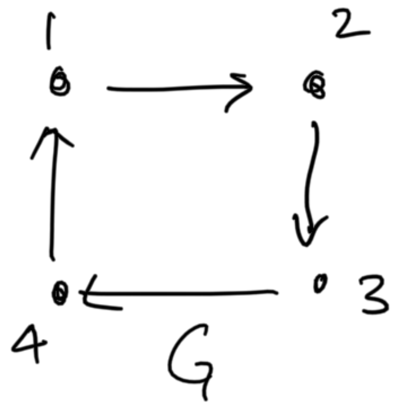
universe of $\mathcal{A}$.

Example Signature $= \{E\}$ (E is binary)

Graph $G = (V, F)$ $V = \{1, 2, 3, 4\}$

$F = \{(1,2), (2,3), (3,4), (4,1)\}$

$\mathcal{G} = (V, E^{\mathcal{G}})$ $E^{\mathcal{G}} = F$.



$\varphi = (\exists x (\exists y \; Exy))$

$\mathcal{G} \models \varphi[\alpha]$ for any α.

$\Rightarrow \mathcal{G} \models (\exists y \; Exy)[\alpha[x \mapsto 1]]$

$\Rightarrow \mathcal{G} \models Exy\,[\alpha[x \mapsto 1][y \mapsto 2]]$

$\Rightarrow$ because $(1,2) \in E^{\mathcal{G}} = \{(1,2), (2,3), (3,4), (4,1)\}$.

Derived Operators

$\varphi \wedge \psi = (\neg((\neg\varphi) \vee (\neg\psi)))$

$\varphi \to \psi = ((\neg\varphi) \vee \psi)$

$\forall x \varphi = (\neg(\exists x (\neg\varphi)))$

$\mathcal{A} \models \forall x \varphi[\alpha]$ iff for every $a \in u(\mathcal{A})$

$\mathcal{A} \models \varphi[\alpha[x \mapsto a]]$.

Precedence $\neg, \vee, \wedge, \to, \exists x, \forall x$.

Scope $\exists x \psi$ ψ is in the scope of $\exists x$.

Bound occurrence $\varphi = \exists x \, \psi$

Every occurrence of x in φ is <u>bound</u>

And any occurrence of y is <u>free</u> if it is not bound.

Example $E x y \land (\exists x (E x y \land P y)) \land Q x$

bound

free

$\equiv E x y \land (\exists z (E z y \land P y)) \land Q x.$

$E x y \land (\exists z \, E z y) \land \exists z \, Q z$

$\equiv E x y \land (\exists z \, E z y) \land (\exists u \, Q u)$

All occurrences of a variable are either free or all bound.

Every bound variable appears in only one quantifier.

Proposition Let φ be a formula and α_1 and α_2 be assignments such that $\alpha_1(x) = \alpha_2(x)$ for every variable x that is free in φ.

Then $\mathcal{A} \models \varphi[\alpha_1]$ iff $\mathcal{A} \models \varphi[\alpha_2]$.

Sentences Formulas φ where every variable is bound.

Proposition Let φ be a sentence. For any pair of assignments α_1, α_2

$$\mathcal{A} \models \varphi[\alpha_1] \text{ iff } \mathcal{A} \models \varphi[\alpha_2]$$

Notation φ is sentence $\mathcal{A} \models \varphi$ if $\mathcal{A} \models \varphi[\alpha]$ for some α.

Satisfiable φ is satisfiable if there is some $\mathcal{A}$ and α s.t $\mathcal{A} \models \varphi[\alpha]$.

Valid φ is valid if for every $\mathcal{A}$ and α

$$\mathcal{A} \models \varphi[\alpha].$$

If φ is valid $\models \varphi$.

Logical Consequence For a set Γ and φ

$$\Gamma \models \varphi \text{ iff } \forall \mathcal{A}, \alpha \quad \underbrace{\mathcal{A} \models \Gamma[\alpha]}_{\forall \psi \in \Gamma,\ \mathcal{A} \models \psi[\alpha]} \Rightarrow \mathcal{A} \models \varphi[\alpha]$$

$\models \varphi$ iff $\emptyset \models \varphi$.

Theory over signature τ is set of sentences over τ.

Inconsistency A theory T is inconsistent if $\exists \varphi .\ \{\varphi, \neg\varphi\} \subseteq T$.

If T is not inconsistent, we say that

If T is not inconsistent, we say that T is consistent.

Completeness T is complete if for every φ $\{\varphi, \neg\varphi\} \cap T \neq \emptyset$.

Theory of $\mathcal{A}$ $Th(\mathcal{A}) = \{\varphi \mid \mathcal{A} \models \varphi\}$

- $Th(\mathcal{A})$ is consistent.
- $Th(\mathcal{A})$ is complete.

If $\mathcal{C}$ is set of structures,

$$Th(\mathcal{C}) = \bigcap_{\mathcal{A} \in \mathcal{C}} Th(\mathcal{A})$$

- $Th(\mathcal{C})$ is consistent?
  - $\mathcal{C}$ is non-empty then $Th(\mathcal{C})$ is consistent.
  - $\mathcal{C} = \emptyset$ then $Th(\emptyset)$ = All sentences and it is inconsistent
- $Th(\mathcal{C})$ is complete?
  - $\mathcal{C}$ to be the set of graphs. $Th(\mathcal{C})$ is not complete.

Γ is a set of sentences

$$Th(\Gamma) = \{\varphi \mid \Gamma \models \varphi\}$$

$$[\![\Gamma]\!] = \{\mathcal{A} \mid \forall \psi \in \Gamma, \mathcal{A} \models \psi\}$$

$$Th(\Gamma) = Th([\![\Gamma]\!])$$