P and NP

Linear Speedup / Compression Theorems

"Constant factors don't matter"

$$\mathrm{DTIME}(T(n)) \subseteq \mathrm{DSPACE}(T(n))$$
$$\mathrm{NTIME}(T(n)) \subseteq \mathrm{NSPACE}(T(n))$$

} Time bounds give an upper bd on memory

$$\mathrm{DTIME}(T(n)) \subseteq \mathrm{NTIME}(T(n))$$
$$\mathrm{DSPACE}(S(n)) \subseteq \mathrm{NSPACE}(S(n))$$

} Det m/c are spl nondet m/c

$$\mathrm{NTIME}(T(n)) \subseteq \mathrm{DSPACE}(T(n))$$
$$\mathrm{NSPACE}(S(n)) \subseteq \mathrm{DTIME}(n \cdot 2^{O(S(n))})$$

Immerman S. Time Hierarchy

$\neq$ $\neq$

$L \to NL \to P \to NP \to \mathrm{PSPACE} = \mathrm{NPSPACE} \to \mathrm{EXP} \to \mathrm{NEXP}$

$NL = coNL$, $P \to coNP \to \mathrm{PSPACE}$

$\neq$

SAVITCH'S

Space Hierarchy

$$coC = \{ A \mid \bar{A} \in C \}$$

Prop: If C is a deterministic complexity class then $coC = C$.

Proof For any $A \in coC$, run algo for $\bar{A}$ and flip answer

P, NP and coNP

Cobham - Edmond's Thesis The class of problems that can be efficiently solved is P.

- Invariance thesis says P is platform invariant
- Most problems in P have running time [illegible]

bounded by a low degree. poly.

- polynomials grow much slower than expo.

Question $P \stackrel{?}{=} NP$

Polynomial Verifiable A problem A is polynomial verifiable if there is deterministic TM V s.t

$$A = \{ x \mid \exists p.\ \langle x, p \rangle \in L(V) \}$$

and $\exists k$ s.t V runs in time $|x|^k$ on input $\langle x, p \rangle$.

WLOG $|p| \leq |x|^k - |x|$

Theorem A is polynomial verifiable iff $A \in NP$.

P : Collection of all problems s.t. you can efficient determine membership of an input / you can find the proof of membership and check correctness efficiently

NP : Collection of problems for which given a proof of the membership of an input you can check it efficiently.

Polytime Reduction Problem A reduces to problem B in polynomial time if there is a function f computable in polynomial time s.t

$\forall x. \quad x \in A$ iff $f(x) \in B$

$A \leq_p B$

Proposition

(a) If $A \leq_p B$ and $B \leq_p C$ then $A \leq_p C$.

(b) If $A \leq_p B$ then $\bar{A} \leq_p \bar{B}$

Theorem Let $\mathcal{C}$ be any complexity Class "above P". And $A \leq_p B$.

If $B \in \mathcal{C}$ then $A \in \mathcal{C}$

If $A \notin \mathcal{C}$ then $B \notin \mathcal{C}$.

Fix $\mathcal{C}$ to be a complexity class containing P.

Hardness A is $\mathcal{C}$-hard if $\forall B \in \mathcal{C} \;\; B \leq_p A$.

Completeness A is $\mathcal{C}$-complete if $A \in \mathcal{C}$ and A is $\mathcal{C}$-hard.

Cook-Levin Theorem Satisfiability problem for propositional logic is NP-complete.

Proof Membership in NP: Given a formula φ and a truth assignment, the verifier will evaluate φ on the truth assignment.

NP-hardness: Let $A \in NP$. Need to show $A \leq_p Sat$.

Let M be a NTM running in time n^l s.t $L(M) = A$.

Given x, we will construct φ_x s.t.

given x, we must construct φ_x s.t.

φ_x is satisfiable iff M accepts x.

Assume M has one work-tape and has a unique accepting state q_a (which is halting) and rejecting state q_r (which is halting) and q_0 is initial state

Propositions Assume $n = |x|$

$n^l \to St(q,i)$ "State M at time i is q".

$n^{l+1} \to InHd(j,i)$ "Input head is on cell j at time i"

$n^{2l} \to WkHd(j,i)$ "Work tape head is on cell j at time i

$n^{2l} \to Wk(b,j,i)$ "Work tape has symbol b in cell j at time i.

Truth assignments to propositions encodes a computation of M on x

$$\varphi_x = \varphi_{init} \wedge \varphi_{consistency} \wedge \varphi_{trans} \wedge \varphi_{accept}.$$

φ_{init} : Truth assignment corresponding to propositions at time 0 correspond to the initial config of M.

$$\varphi_{init} = St(q_0, 0) \wedge InHd(0,0) \wedge WkHd(0,0) \wedge \bigwedge_{j=0}^{n^l-1} Wk(\sqcup, j, 0)$$

$$\varphi_{\ldots} = \bigwedge^{n^l-1} \Big[Unq(St(q, i), St(q, i), \ldots St(q, i)) \ldots$$

$$T_{consistency} = \bigwedge_{i=0} \Big[\mathrm{Uniq}\big(St(q_0,i), St(q_1,i) \ldots St(q_k,i)\big)$$

$$\wedge\ \mathrm{Uniq}\big(InHd(0,i), \ldots InHd(n-1,i)\big)$$

$$\wedge\ \mathrm{Uniq}\big(WkHd(0,i) \ldots WkHd(n^{\ell}-1,i)\big)$$

$$\bigwedge_{j=0}^{n^{\ell}-1} \mathrm{Uniq}\big(Wk(b_1, j,i), Wk(b_2,j,i) \ldots\big) \Big]$$

$\mathrm{Uniq}(p_1 \ldots p_k)$ – Exactly one p_i is true.