Rules of proof system

$$\frac{\Gamma}{\varphi} \quad \begin{array}{l} \Gamma \to \text{set of formulas} \\ \varphi \to \text{formula} \end{array}$$

$$\frac{\varphi}{\varphi} \quad \Big| \quad \frac{}{\varphi} \quad \text{Axioms}$$

Assume Formulas built up using propositions $\bot$ (false) and $\to$ (implication)

$\neg\varphi \equiv (\varphi \to \bot)$

$\varphi \vee \psi \equiv ((\varphi \to \bot) \to \psi)$

Frege Style Proof System

$$\frac{}{\varphi \to (\psi \to \varphi)} \qquad \frac{}{(\varphi \to (\psi \to \rho)) \to ((\varphi \to \psi) \to (\varphi \to \rho))}$$

$$\frac{}{(\underbrace{(\varphi \to \bot) \to \bot}_{\neg\neg\varphi}) \to \varphi} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi} \text{ [Modus Ponens]}$$

Example $p \to (q \to p)$ is an axiom

Proofs A proof of φ from assumptions Γ is a sequence of formulas $C_1, C_2 \ldots C_n$ s.t.

(a) $C_n = \varphi$

(b) $\forall i$ $C_i \in \Gamma$ or C_i is an axiom or there $j, k < i$ s.t. C_i follows from C_j & C_k

by Modus Ponens.

$$\frac{}{\varphi \to (\psi \to \varphi)} \qquad \frac{}{(\varphi \to (\psi \to \rho)) \to ((\varphi \to \psi) \to (\varphi \to \rho))}$$

$$\frac{}{((\varphi \to \bot) \to \bot) \to \varphi} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}$$

Example: Prove p from assumption $\bot$

1	$\bot \to ((p \to \bot) \to \bot)$	(Axiom 1)
2.	$\bot$	($\bot \in \Gamma$)
3.	$(p \to \bot) \to \bot$	(MP)
4.	$((p \to \bot) \to \bot) \to p$	(Axiom 3)
5.	p	(MP, 3,4)

Notation $\Gamma \vdash \varphi$ if there is a proof of φ from assumptions Γ. When $\Gamma = \emptyset$, $\vdash \varphi$

Logical consequence $\Gamma \models \varphi$

Soundness If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

Completeness If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$

Tautology $\models \varphi \Rightarrow \vdash \varphi$

Resolution

Easily mechanizable

Method to establish refutations

Conjunctive Normal Form

1. **Literal** is either a proposition or its negation e.g. $p, \neg q$
2. **Clause** is a disjunction of literals e.g. $p \vee \neg q \vee r$. Clauses will be represented as a set of literals. e.g. $\{p, \neg q, r\}$

   A valuation v satisfies a clause C if some literal $l \in C$ $v[\![l]\!] = T$
3. **Empty clause** $\{\}$. Unsatisfiable
4. **CNF** are conjunctions of clauses e.g. $(p \vee \neg q \vee r) \wedge (\neg q \vee r)$. Represent this as a set of clauses e.g. $\{\{p, \neg q, r\}, \{\neg q, r\}\}$

Proposition Every formula is logically equivalent to a formula in CNF.

Resolution Rule
$$\frac{C \cup \{p\} \quad D \cup \{\neg p\}}{C \cup D}$$
Resolvent w.r.t p of $C \cup \{p\}$ & $D \cup \{\neg p\}$

Example
$$\frac{\{p, \neg q\}, \quad \{\neg p, q\}}{\{\neg q, q\}} \qquad \frac{\{p, \neg q\} \quad \{\neg p, q\}}{\{p, \neg p\}}$$

Refutation A refutation of a set of clauses Γ is a sequence of clauses $C_1, C_2 \ldots C_n$ such that

1. $C_n = \{\}$
2. $\forall i$ either $C_i \in \Gamma$ or
   C_i is resolvent of C_j and C_k $j,k < i$

Example $\Gamma = \{\{p,q\}, \{\neg p, r\}, \{\neg q, r\}, \{\neg r\}\}$

Refutation of Γ

1. $\{p, q\}$ $(\in \Gamma)$
2. $\{\neg p, r\}$ $(\in \Gamma)$
3. $\{q, r\}$ (Resolvent of 1 & 2)
4. $\{\neg q, r\}$ $(\in \Gamma)$
5. $\{r\}$ (Resolvent of 3, 4)
6. $\{\neg r\}$ $(\in \Gamma)$
7. $\{\}$ (Resolvent of 5, 6)

Soundness If a set of clauses Γ has a refutation then Γ is unsatisfiable.

Completeness If Γ is unsatisfiable then Γ has a refutation

φ is valid iff $\neg\varphi$ is unsatisfiable

φ is valid using resolution
- Convert $\neg\varphi$ into CNF
- Construct a refutation for $\neg\varphi$